Probability Distribution

Random Variable or Variate

If a variable X ia said to be a random variable if corresponding to each value x_i of a variable x, there exists a probability p_i.

e.g. A fair die has six faces. When a fair die is thrown, then there are six possible events i.e. 1,2,3,4,5,6. These are called values of the variable.

Discrete Random Variate

If a random variable takes a finite set of values, then it is called a discrete random variate.

e.g. If a variable x considered number of heads when a coin is tossed twice, then x takes values 0,1,2.

Continuous Random Variate

If a random variable takes infinite number of values or any value within a certain interval is called continuous random variable.

e.g. If x represents height of school children in cm. then it may take any value in the interval $130 \le x \le 180$ cm..

Probability Distribution

If a variate x takes values $x_1, x_2, x_3, ---, x_i, ---x_n$ having respective probabilities $p_1, p_2, p_3, ---, p_i, ---p_n$ and $p_1 + p_2 + p_3 + ---, +p_i + --- + p_n = 1$, then the probability distribution of a random variable x is given as follows.

x_i	x_1	x_2	x_3	--	x_i	--	x_n
p_i	p_1	p_2	p_3	--	p_i	--	p_n

We study discrete probability distribution and continuous probability distribution. There are many distributions. In this chapter, we study following distributions.

1) Binomial distribution
2) Poisson distribution
3) Normal distribution

Binomial Distribution

This is discrete probability distribution in which we are concerned with success or failure of an event. If p is the probability of success and $q = 1 - p$ is the probability of failure, then the probability of x successes in a series of n trials is given as follows.

$$p(x) = {}^nC_x p^x q^{n-x}, \text{ where } x = 0,1,2,---x,---n$$

The probability of the number of successes so obtained is called binomial distribution because the probabilities are the successive terms in the expansion of the binomial $(q+p)^n$.

Note:

1) Mean of binomial distribution is np. $\therefore$ Mean $= \mu = np$
2) Standard Deviation $= \sigma = \sqrt{npq}$
3) Variance $= \sigma^2 = npq$
4) The probability that an event happens exactly x times i.e. x successes is $p(x) = {}^nC_x p^x q^{n-x}$.
5) The probability that an event happens at least x times is $p(\ge x)$.
6) The probability that an event happens at most x times is $p(\le x)$
7) If there are N sets each containing n trials, then the expected frequencies corresponding to $x = 0,1,2,---,n$ successes is given as follows.

$$\text{Expected Frequencies} = f(x) = p(x) = N.{}^nC_x p^x q^{n-x} = \text{N.}(q+p)^n$$

Examples

1) A person fires 10 shots at target. The probability that any shot will hit the target is $\frac{3}{5}$. Find the probability that the target is hit exactly 5 times.[p(5) = 0.2007] **[BTE 2017]**

2) In 200 sets of tosses of 5 fair coins , in how many ways you can expect
i) at least two heads. ii) at the most two heads. **[BTE 2016]**

3) If 30% of the bulbs produced are defective, find the probability that out of 4 bulbs selected
i) one is defective
ii) at the most two are defective. **[BTE 2015]**

Examples for Tutorial

1) The probability that a person is a swimmer is 2/5. What is the probability that out of 4 persons contacted at random
a) exactly 1, b) at least 1 is a swimmer.
[a) 125/216 b) 544/625]

2) 8 fair coins are tossed simultaneously. Determine the probability of getting at least 6 heads. **[37/256]**

3) If 10% of the rivets produced by a machine are defective. Determine the probability that out of 5 rives chosen at random
a) none, b) not more than 2 are defective.
[a) 0.59049 b) 0.72905]

Poisson Distribution

Poisson distribution is a discrete probability distribution. It is the limiting form of Binomial distribution when
i) the number of trials n is very large i.e. $n \to \infty$
ii) the probability p of success is very small i.e. $p \to 0$
iii) the mean np is a finite constant
Poisson distribution is given by,

$$P(X = x) = \frac{e^{-m} m^{x}}{x!}$$

The random variable x is a Poisson variate and m is a parameter of the distribution.
Note:
1) The mean of a Poisson distribution is $m = np$.
2) The variance of a Poisson distribution is $V = m$.
3) The standard deviation of a Poisson distribution is $\text{S.D.} = \sqrt{m}$.

Examples

1) Assuming that the probability of a fatal accident during the year is $\frac{1}{1200}$. Calculate the probability that in a factory employing 300 workers, there will be at least two fatal accidents in a year. $[e^{-0.25} = 0.7788]$ **[BTE 2017]**

2) A company manufactures electric motors. The probability of an electric motor is defedtive is 0.01. What is the probability that a sample of 300 electric motors will contain exactly 5 defective motors? $[e^{-3} = 0.0498]$ **[BTE 2017]**

3) If 5% of the electric bulbs manufactured by a company are defective. Use Poisson distribution to find the probability that in a sample of 100 bulbs.
i) None is defective.
ii) Five bulbs are defective. **[BTE 2016]**

Examples for Tutorial

1) Fit a Poisson distribution for the following distribution. **[BTE2015]**

x_i	20	30	40	50	60	70
f_i	8	12	20	10	6	4

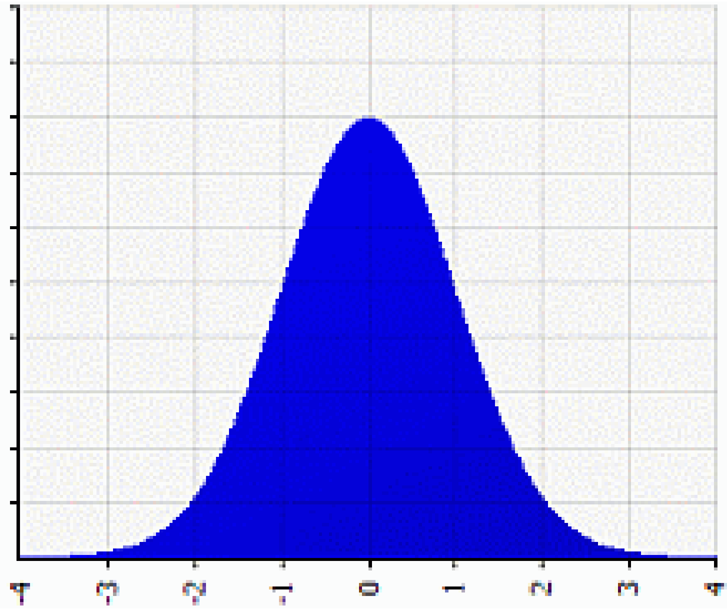
$$\left[P(X = x) = \frac{e^{-41}(41)^x}{x!} \right]$$

2) Using Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well-shuffled cards at least one in 104 consecutive trials. **[BTE 2013]**
[0.865]

3) If 5% of the items manufactured by a company are defective. Use Poisson distribution to find the probability that in a sample of 100 items
i) none is defective.
ii) five items are defective. [Given $e^{-5} = 0.007$] **[BTE 2014]**
[i) P(0) = 0.007, ii) P(5) = 0.1823]

Normal Distribution

Normal distribution is a continuous probability distribution. This distribution was discovered by De Moivre's in 1733. Normal curve for this distribution is as shown in the following figure.



Let $z = \frac{x - \mu}{\sigma}$ be a variate where x is a binomial variate with mean $\mu = np$ and standard deviation $\sigma = \sqrt{npq}$. z is a variate with mean zero and variance unity. It is called **Standard Normal Variable (S.N.V.)**

The equation of normal curve is given by

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad , \quad -\infty < z < \infty$$

Note:

1) The area under the normal curve is always 1.
2) The area under the normal curve is obtained from a statistical table.
3) If $c' = \frac{c - \mu}{\sigma}$ and $d' = \frac{d - \mu}{\sigma}$ are standardized values of c and d, then

$$P(c < x < d) = \text{Probability that } x \text{ lies between } c \text{ and } d$$

$$= \phi\left(\frac{d-\mu}{\sigma}\right) - \phi\left(\frac{c-\mu}{\sigma}\right)$$

$$= \phi(d') - \phi(c')$$

Properties of Normal Curve

1) The normal curve is bell-shaped and it is symmetrical about the mean.
2) Mean, median and mode of a normal distribution are equal.
3) The ordinate at the mean cuts the curve in two equal halves.

Examples

1) In a test of 2000 electric bulbs, it was found that the life of a particular make was normally distributed with average life of 2040 hours and standard deviation of 60 hours. Estimate the number of bulbs likely to burn for

i) between 1920 hours and 2160 hours

ii) more than 2150 hours.

Given that $A(2) = 0.4772, \ A(1.83) = 0.4664$ **[BTE2016,17]**

2) In a sample of 1000 cases, the mean of certain test is 14 and standard deviation is 2.5. Assuming the distribution is normal. Find

i) Hoe many students score between 12 and 15?

ii) How many students score above 18?

[Given A(0.8) = 0.2881, A(0.4) = 0.1554, A(1.6) = 0.4452] **[BTE 2013,14,16]**

3) I.Q.s are normally distributed with mean 100 and standard deviation 15. Find the probability that a randomly selected person has

i) An I.Q. more than 130. [Ans. 0.0228]

ii) An I.Q. between 85 and 115. [Ans. 0.6826]

[z = 2, Area = 0.4772, z = 1, Area = 0.3413] **[BTE 2015, 2017]**

Examples for Tutorial

1) The mean weight of 500 students at a certain college is 50 kg. and standard deviation is 6 kg. Assuming the weights are normally distributed, find the number of students weighing between 40 kg. and 50 kg. [Given A(1.67) = 0.4525] **[BTE 2014, 15,16]**
[226]

2) The weekly wages of workers in a cetain factory was found to be normally distributed with mean Rs. 500 and standard deviation is equal to Rs. 50. There are 228 persons getting Rs. 600. Find the number of workers in a factory. **[10,000]**

3) A random saple of 200 screws is drawn from a population which represent the size of screw. If a sample is distributed normally with mean 3.15 cm. and standard deviation 0.025 cm.then find the expected number of screws whose size falls between 3.12 and 3.2 cm. **[BTE'90] [172]**