Partial Fractions

1) Rational Fraction: A fraction of the form R(x) = P(x)/Q(x) is called rational fraction.

2) **Proper Fractions**: If the degree of denominator is greater than the degree of numerator in the rational fraction, then it is called proper fraction.

3) **Improper Fractions**: If the degree of numerator is greater than or equal to the degree of denominator in the rational fraction, then it is called improper fraction.

4) **Partial Fraction**: The method of expressing proper fraction as a sum or difference of proper fractions having linear or quadratic factors in the denominator is called partial fraction.

We study following cases of partial fractions.

Case I) **Non-repeated linear factors occur in the denominator:** If the denominator Q(x) has all linear factors, then

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(x+a)(x+b)(x+c)---(x+d)}=\frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{x+c}+---+\frac{Z}{x+z}.$$

Case II) **Repeated linear factors occur in the denominator**: If the denominator Q(x) has repeated linear factors occur n times, then

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(x+a)^n}=\frac{A}{(x+a)}+\frac{B}{(x+a)^2}+\frac{C}{(x+a)^3}+----+\frac{Z}{(x+a)^n}.$$

Case III) **Irreducible quadratic factors occur in the denominator:**

If the denominator Q(x) has irreducible quadratic factors, then

$$\frac{P(x)}{Q(x)}=\frac{Ax+B}{ax^2+bx+c}.$$

Case IV) **Repeated irreducible quadratic factors occur in the denominator**:

If the denominator Q(x) has irreducible repeated quadratic factors occur n times, then

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(ax^2+bx+c)^n}=\frac{A}{(ax^2+bx+c)}+\frac{B}{(ax^2+bx+c)^2}+\frac{C}{(ax^2+bx+c)^3}+---+\frac{Z}{(ax^2+bx+c)^n}.$$

Note: If the degree of numerator is greater or equal to the degree of denominator, then divide numerator by denominator and use above cases for partial fractions.

Examples: Resolve into partial fraction **: (Non-repeated linear factors occur in the denominator)**

Examples	**Answers**
1) $\frac{x+3}{(x^2-1)(x+5)}$ **[BTE 2017]**	$\frac{x+3}{(x^2-1)(x+5)}=\frac{1/3}{x-1}+\frac{-1/4}{x+1}+\frac{-1/12}{x+5}$
2) $\frac{1}{x^2+3x+2}$ **[BTE 2015]**	$\frac{1}{x^2+3x+2}=\frac{1}{x+1}+\frac{-1}{x+2}$
3) $\frac{1}{x^3+3x^2+2x}$ **[BTE 2016]**	$\frac{1}{x^3+3x^2+2x}=\frac{1/2}{x}+\frac{-1}{x+1}-\frac{1/2}{x+2}$
4) $\frac{x^2+1}{(x^2+2)(x^2+3)}$ **[IoPE 2017]**	$\left[\frac{x^2+1}{(x^2+2)(x^2+3)}=\frac{-1}{x^2+2}+\frac{2}{x^2+3}\right]$

Examples for Tutorial

Examples	Answers
1) $\frac{1}{x(x+1)}$ **[BTE 2016]**	$[\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}]$
2) $\frac{x+4}{x(x+1)}$ **[BTE 2017]**	$[\frac{x+4}{x(x+1)}=\frac{4}{x}+\frac{-3}{x+1}]$
3) $\frac{1}{x^2-x}$ **[BTE 2017]**	$\left[\frac{1}{x^2-x}=\frac{-1}{x}+\frac{1}{x-1}\right]$
4) $\frac{3x-1}{(x-4)(2x+1)(x-1)}$ **[BTE2015, 2016]**	$\left[\frac{3x-1}{(x-4)(2x+1)(x-1)}=\frac{11/27}{x-4}-\frac{10/27}{2x+1}-\frac{2/9}{x-1}\right]$
5) $\frac{x-5}{x(x+3)(x-2)}$ **[BTE 2016]**	$\left[\frac{x-5}{x(x+3)(x-2)}=\frac{5/6}{x}-\frac{8/15}{x+3}-\frac{3/10}{x-2}\right]$
6) $\frac{2}{x^2+x-2}$ **[BTE 2016]**	$\left[\frac{2}{x^2+x-2}=\frac{2/3}{x-1}-\frac{2/3}{x+2}\right]$

Examples: Resolve into partial fraction **: [Degree of numerator and denominator is same and degree of numerator is greater than the degree of denominator]**

Examples	Answers
1) $\frac{x^2+1}{x^2-1}$ **[IoPE 2015]**	$\left[\frac{x^2+1}{x^2-1}=1+\frac{-1}{(x+1)}+\frac{1}{(x-1)}\right]$
2) $\frac{x^3+x}{x^2-9}$ **[BTE 2017]**	$\left[\frac{x^3+x}{x^2-9}=x+\frac{5}{x+3}+\frac{5}{x-3}\right]$

Examples for Tutorial

Examples	Answers
1) $\frac{x^3+x}{x^2-4}$ **[BTE 2016]**	$\frac{x^3+x}{x^2-4}=x+\frac{5/2}{x-2}+\frac{5/2}{x+2}$
2) $\frac{x^3+1}{x^2+2x}$ **[BTE 2015]**	$\frac{x^3+1}{x^2+2x}=(x-2)+\frac{1/2}{x}+\frac{7/2}{x+2}$

Examples: Resolve into partial fraction **: [Repeated linear factors occur in the denominator]**

Examples	**Answers**
1) $\frac{2x+3}{x^2(x-1)}$ **[BTE2015]**	$\left[\frac{2x+3}{x^2(x-1)}=\frac{-5}{x}+\frac{-3}{x^2}+\frac{5}{x+1}\right]$
2) $\frac{2x-3}{(x+1)(x^2-1)}$ **[BTE 2017]**	$\left[\frac{2x-3}{(x+1)(x^2-1)}=\frac{-1/4}{x-1}+\frac{1/4}{x+1}+\frac{5/2}{(x^2+1)}\right]$

Examples for Tutorial

Examples	**Answers**
1) $\frac{x^2}{(x+1)(x+2)^2}$ **[BTE 2017]**	$\left[\frac{x^2}{(x+1)(x+2)^2}=\frac{1}{x+1}-\frac{4}{(x+2)^2}\right]$
2) $\frac{2x-3}{(x+1)(x^2-1)}$ **[BTE 2017]**	$\left[\frac{2x-3}{(x+1)(x^2-1)}=\frac{-1/4}{x-1}+\frac{1/4}{x+1}+\frac{5/2}{(x^2+1)}\right]$
3) $\frac{3x+6}{(x+6)(x+2)^2}$ **[IoPE2015]**	$\left[\frac{3x+6}{(x+6)(x+2)^2}=\frac{-4/5}{(x+6)}+\frac{23/30}{(x+2)}+0=\frac{-4/5}{(x+6)}+\frac{23/30}{(x+2)}\right]$
4) $\frac{2x+1}{x^2(x+1)}$ **[BTE 2016]**	$\left[\frac{2x+1}{x^2(x+1)}=\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x+1}\right]$

Examples: Resolve into partial fraction **: [Quadratic factors occur in the denominator]**

Examples	**Answers**
Resolve into partial fractions.	
1) $\frac{2x-3}{(x+1)(x^2+4)}$ **[BTE2015, 2016]**	$\left[\frac{2x-3}{(x+1)(x^2+4)}=\frac{-1}{x+1}+\frac{(1)x+1}{x^2+4}\right]$
2) $\frac{x}{(x^3+1)}$ **[BTE 2015]**	$\left[\frac{x}{(x^3+1)}=\frac{-1/3}{x+1}+\frac{(1/3)x+(1/3)}{x^2-x+1}\right]$
3) $\frac{x^2-2x+3}{x^3+x}$ **[BTE 2017]**	$\left[\frac{x^2-2x+3}{x^3+x}=\frac{3}{x}+\frac{-2x-2}{x^2+1}\right]$

Examples for Tutorial

Examples	Answers
1) $\frac{x^2+23x}{(x+3)(x^2+1)}$ **[BTE 2016]**	$\left[\frac{x^2+23x}{(x+3)(x^2+1)}=\frac{-6}{x+3}+\frac{7x+2}{x^2+1}\right]$
2) $\frac{x^2+1}{(x+1)(x^2+4)}$ **[BTE 2017]**	$\left[\frac{x^2+1}{(x+1)(x^2+4)}=\frac{2/5}{x+1}+\frac{(3/5)x-(3/5)}{x^2+4}\right]$
3) $\frac{x^2-x+3}{(x-2)(x^2+1)}$ **[BTE 2017]**	$\left[\frac{x^2-x+3}{(x-2)(x^2+1)}=\frac{1}{x-2}+\frac{0x-1}{x^2+1}\right]$