Numerical Methods
Numerical Solution of Algebraic Equations

In numerical analysis, a numerical method is a mathematical tool designed to solve numerical problems. Such problems are from the natural sciences, social sciences, engineering, medicine, and business.

Iterarive Methods

Iterative methods are used when we cannot find the solution of algebraic equation by direct method. We find the approximate root of non-linear algebraic equation by iterative methods. This approximate solution is close to exact solution but never equal to exact solution. The repeatation of steps again and again to find approximate root of equation in the following methods are called iterative methods and each step is called iteration.
We study following methods to find root of non-linear algebraic equation.
1) The Bisection Method
2) False Position Method or Regula Falsi Method
3) Newton Raphson Method

The Bisection Method

This is the simplest method of all methods.
Let $f(x)=0$ be an algebraic equation. For any two real numbers m and n, we find $f(m)$ and $f(n)$.
If $f(m)<0$ and $f(n)>0$ or $f(m)>0$ and $f(n)<0$ that is, $f(m)$ and $f(n)$ have opposite signs, then the root of the equation $f(x)=0$ lies in the interval (m,n),

Steps to solve examples

1) Find two values say m and n such that $m<n$ and $m<n$
$f(m)<0,\ f(n)>0$ or $f(m)>0,\ f(n)<0$.

2) Find the midpoint of m and n say x.

3) x is root of function if $f(x)=0$ otherwise follow the next step.

4) Divide the interval [m, n].

5) Repeat above two steps until $f(x)=0$.

Examples

1) Using Bisection method, find the approximate root of $x^3-x-4=0$.
(Three iterations only) **[BTE2015, 2017]**
2) Show that the root of the equation $xe^x=1$ lies between 0 and 1. $[f(0)=-1<0,\ f(2)=2>0]$

Examples for Tutorial

1) Show that the root of the equation $x^3-9x+1=0$ lies between 2 and 3. $[f(2)=-9<0,\ f(3)=1>0]$

2) Find the root of the equation $x^2-16=0$ by using Bisection method. $[x_1=4]$

3) Using Bisection method, find the root of the equation $x^3-x-1=0$.
(Two iterations only) $[x=1.25]$

Regula Falsi Method or False Position Method

Bisection method is slow. So, we need faster method to reduce number of iterations. Regula falsi method is faster method.

Steps to solve the example

1) Choose two trial values $x = a$ and $x = b$. Find $f(a)$ and $f(b)$.

2) If $f(a)$ and $f(b)$ have opposite signs, then the first approximate root of $f(x)$ is given by,

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

3) Find $f(x_1)$. If $f(x_1) \neq 0$, we have two possible cases.

Case 1) If $f(a)$ and $f(x_1)$ are of opposite signs, then the appropriate root is given by,

$$x_2 = \frac{af(x_1) - x_1 f(a)}{f(x_1) - f(a)}$$

Case 2) If $f(x_1)$ and $f(b)$ are of opposite signs, then the appropriate root is given by,

$$x_2 = \frac{x_1 f(b) - bf(x_1)}{f(b) - f(x_1)}$$

Repeat this process till you get the root close to the real root.

Example

1) Find the root of by using Regula-Falsi method $xe^x - 3 = 0$ (Three iterations only) **[BTE2017]**

[$f(1) < 0$, $f(2) > 0$, $x_1 = 1.0234$, $x_2 = 1.0358$, $x_3 = 1.0425$]

Examples for Tutorial

1) Find the root of $x^2 - x - 1 = 0$ by using Regula-Falsi method up to third approximation. **[BTE2015]** [$f(1) < 0$, $f(2) > 0$, $x_1 = 1.5$, $x_2 = 1.6$, $x_3 = 2.3077$]

2) Using false position method, find the root of the equation $x^2 + x - 3 = 0$ in the interval (1,2) by performing two iterations. [$f(1) < 0$, $f(2) > 0$, $x_1 = 1.25$, $x_2 = 1.294$]

Newton Raphson Method

Let m be a real root of the equation $f(x)=0$, which is near to the point $x=a$. Then Newton Raphson formula for iteration is,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

Steps to solve examples

1) Choose two trial values $x=a$ and $x=b$. Find $f(a)$ and $f(b)$ using $f(x)$ from $f(x)=0$.

2) If $f(a)$ and $f(b)$ have opposite signs, then the root of $f(x)=0$ lies in the interval $[a,b]$. Estimate the initial or starting approximation to the root as x_0 from the function.

3) The first approximate root is given by,

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}.$$

4) Find $f(x_1)$.

i) If $f(x_1)=0$, then x_1 is required root.

ii) If $f(x_1)\neq 0$, then the second approximate root is given by,

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

5) Repeat the process till you get the root close to the real root.

Examples

1) Find the root of the equation $x^3+x-1=0$ by Newton-Raphson method up to three iterations only. **[BTE2016]**

2) Evaluate $\sqrt[3]{100}$ by using Newton-Raphson method. (Three iterations only). **[BTE2017]**

Examples for Tutorial

1) Solve by Newton Raphson method, $x^3+2x-20=0$.(Three iterations only). **[BTE2017]**

2) Show that there exist a root of the equation $x^3+2x^2-8=0$ between 1 and 2. **[BTE2017]** $[f(1)=-5<0, f(2)>0]$

3) Show that there exist the root of the equation $x^3-5x-11=0$ between 2 and 3. **[BTE2016]**

4) Show that the root of the equation $x^3-4x+1=0$ in (1,2) and find it by using Newton Raphson method performing two iterations. **[BTE2015]**

5) Find the root of the equation $x^2-4x-6=0$ near to 5 by using Neton Raphson method. (Three iterations only).. **[BTE2017]**

6) Prove that the root of the equation $x^3-x-4=0$ lies between 0 and 2. **[BTE2017]** $[f(0)=-4<0, f(2)=2>0]$

7) Prove that the root of the equation $x^3-9x+1=0$ lies between 2 and 3. **[BTE2015]** $[f(2)=-9<0, f(3)=1>0]$

8) Using Newton-Raphson method to find (i) $\sqrt{10}$ (ii) $\sqrt[3]{20}$ correct to three decimal places.**[BTE2015,2016]**