LIMIT

1) The dictionary meaning of limit is boundary.
2) $x \to a$ is read as x tends to a.
3) $x \to 2$ means the value of x is nearest to the number 2 but not exactly 2. It means the value of x may be 1.9, 1.95 or 2.1 etc.

Limit of a function:
If f(x) is a function then the limit of f(x) as $x \to a$ is written as $\lim_{x \to a} f(x) = l$ where l is the value of limit of a function.

Algebra of Limits:
If $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)}$ where $g(a) \neq 0$, then in this case limit as $x \to a$ can be obtained by substitution of x by a.
Let f(x) and g(x) be two functions such that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then

1) Addition : $\lim_{x \to a}\left[f(x) + g(x)\right] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = l + m$

2) Subtraction: $\lim_{x \to a}\left[f(x) - g(x)\right] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = l - m$

3) Multiplication: $\lim_{x \to a}\left[f(x).g(x)\right] = \lim_{x \to a} f(x).\lim_{x \to a} g(x) = l.m$

4) Division: If $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}$ where $m \neq 0$

5) If k is constant, then $\lim_{x \to a} k.f(x) = k \lim_{x \to a} f(x) = k.l$

Methods of finding limits:
1) Direct Method:
If $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)}$ where $g(a) \neq 0$, then in this case limit as $x \to a$ can be obtained by substitution of x by a.

For example:
$$\lim_{x \to 1}\left[\frac{x+2}{x-5}\right] = \frac{\lim_{x \to 1}(x+2)}{\lim_{x \to 1}(x-5)} = \frac{1+2}{1-5} = \frac{3}{-4} = -\frac{3}{4}$$

Examples

1) $\lim_{x \to 3} \frac{\sqrt{x} + \sqrt{3}}{x+3}$ **[B.T.E.2017]** 2) $\lim_{x \to \pi} \frac{\sin x}{\cos^2 x}$

Examples for Tutorial

1) $\lim_{x \to 1} \frac{3x^2 + 5x}{2x+3}$ **[8/5]** 2) $\lim_{x \to 0} \frac{2x^2 + 3x + 4}{x^2 + 2x + 1}$ **[4]** 3) $\lim_{x \to 1} \frac{x^2 + 2x + 5}{x+1}$ **[B.T.E.2015] [4]**

Method of factorization

If the direct method fails, i.e. if $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\frac{f(a)}{g(a)}=\frac{0}{0}$, which is an indeterminate form, then $(x-a)$ is a common factor in the numerator and denominator. We cancel this common factor and substitute $x=a$ in the remaining expression.

Examples

1) $\lim_{x\to 2}\frac{x^3-x^2-8x+12}{x^3-12x+16}$ **[IoPE 2012]**

2) $\lim_{x\to 3}\frac{x^3-7x^2+15x-9}{x^3-4x^2-3x+18}$ **[B.T.E.2015]**

3) $\lim_{x\to 2}\frac{x^3-3x^2+4}{x^4-8x^2+16}$ **[B.T.E.'90]**

Examples for Tutorial

1) $\lim_{x\to 3}\frac{x^2-x-6}{x^3-3x^2+x-3}$ **[B.T.E.2016], [1/2]**

2) $\lim_{x\to 5}\frac{x^3-125}{x^2-7x+10}$ **[B.T.E.'86], [25]**

3) $\lim_{x\to 5}\frac{x^2-9x+20}{x^2-6x+5}$ **[B.T.E.2017] [1/4]**

4) $\lim_{x\to 1}\left(\frac{x^3-1}{x-1}\right)$ **[B.T.E.2016] [3]**

5) $\lim_{x\to 3}\frac{x^2-9}{x-3}$ **[B.T.E.2015] [6]**

Method of rationalization

If the numerator or denominator or both contain irrational function of the type $\sqrt{a}\pm\sqrt{b}$ and we get an indeterminate form after substituting the value of limit in the given function, then rationalize numerator or denominator or both in which irrational function occurs.

Examples

1) $\lim_{x\to 2}\frac{\sqrt{6-x}-\sqrt{2+x}}{5x-10}$ **[B.T.E.May'92]**

2) $\lim_{x\to 2}\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$ **[B.T.E.2016]**

3) $\lim_{x\to 3}\frac{\sqrt{x^2+7}-4}{\sqrt{2x+3}-\sqrt{4x-3}}$ **[IoPE 2006]**

Examples for Tutorial

1) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$, **[B.T.E. 2016]** **[1] Solved**

2) $\lim_{x\to 0}\frac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{x^2}$, **[B.T.E.'90]** $\left[1/a\right]$

3) $\lim_{x\to 0}\frac{x}{\sqrt{a+x}-\sqrt{a-x}}$, **[B.T.E.'91]** $\left[\sqrt{a}\right]$

4) $\lim_{x\to 2}\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$, **[-8]**

5) $\lim_{x\to 1}\frac{\sqrt{x+8}-\sqrt{8x+1}}{\sqrt{5-x}-\sqrt{7x-3}}$, **[7/12]**

6) $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}$ **[1]**

Type : $\lim_{x\to a}\frac{x^n-a^n}{x-a}=n\,a^{n-1}$

Examples

1) $\lim_{x\to 2}\frac{x\sqrt{x}-2\sqrt{2}}{x-2}$ **[B.T.E.'92]**

2) $\lim_{x\to 2}\frac{x^5-32}{x^3-8}$ **[IoPE 2016, B.T.E.'85]**

3) $\lim_{x\to 1/2}\frac{8x^3-1}{16x^4-1}$ **[IoPE 2014, B.T.E.'96]**

4) $\lim_{x\to -3}\frac{x^7+2127}{x^5+243}$

Examples for Tutorial

1) $\lim_{x\to 3}\frac{x^4-81}{x-3}$ **[108]**

2) $\lim_{x\to a}\frac{x^{1/5}-a^{1/5}}{x-a}$ $\left[a^{-4/5}/5\right]$

3) $\lim_{x\to 2}\frac{x^{10}-1024}{x^5-32}$ **[B.T.E.2016]** **[64]**

4) $\lim_{x\to -1/2}\frac{128x^7+1}{8x^3+1}$ **[7/3]**

Limit of the function f(x) when $x\to\infty$:

We use following procedure when $x\to\infty$.

1) Divide numerator and denominator by highest value of x available in the example.

2) If denominator = 1, then rationalize the numerator and bring a factor in the denominator.

3) As $x\to\infty$, then $\frac{1}{x^p}\to 0$ where p > 0.

EXAMPLES

1) $\lim_{x\to\infty}\frac{(3x+5)(9x-2)(8x+3)}{5x^3-7x+9}$ **[IoPE 2016]**

2) $\lim_{x\to\infty}\frac{2x^2-3x+5}{3x^2+27x-29}$ **[IoPE 2012]**

3) $\lim_{n\to\infty}\frac{1^2+2^2+-----+n^2}{n^3}$. **[IoPE 2014]**

4) $\lim_{x\to\infty}\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right)$ **[B.T.E. 2016]**

Examples for Tutorial

1) $\lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2-5x+7}$ **[B.T.E.'79] [3/2]** 2) $\lim_{x\to\infty}\frac{x+\sqrt{x^2+4}}{x+3}$ **[2]** 3) $\lim_{n\to\infty}\frac{n^2}{1+2+3+-----+n}$ **[2]**

4) $\lim_{n\to\infty}\frac{1^3+2^3+-----+n^3}{(n-1)^4}$ **[1/4]** 5) $\lim_{x\to\infty}(\sqrt{x^2+x+1}-x)$ **[B.T.E.2015]** **[1/2]**

Trigonometric Limits

I) By factorization

Sometimes we evaluate the limits by factorizing numerator or denominator or both. Cancelling the common factor in the numerator and denominator and we use the following results:

1) $\lim_{x\to a}\sin x=\sin a$, 2) $\lim_{x\to a}\cos x=\cos a$, 3) $\lim_{x\to a}\tan x=\tan a$.

Examples:

1) $\lim_{x\to\pi}\frac{1+\cos^3 x}{\sin^2 x}$ **[B.T.E.'88]** 2) $\lim_{x\to\pi/4}\frac{\sec^2 x-2}{\tan x-1}$ **[IoPE2009, B.T.E.'87]**

3) $\lim_{\theta\to\frac{\pi}{4}}\frac{2-\sec^2\theta}{1-\tan\theta}$ **[B.T.E.2016]**

Examples for Tutorial

1) $\lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{\sin^2 x}$ **[1/4]** 2) $\lim_{\theta\to\pi/6}\frac{\cot^2\theta-3}{\operatorname{cosec}\theta-2}$ **[4]** 3) $\lim_{x\to\pi/4}\frac{\sin^2 x-\cos^2 x}{1-\tan x}$ **[-1]**

4) $\lim_{x\to\pi/3}\frac{3\sin^2 x-\cos^2 x-2}{1-2\cos x}$ **[2]**

II) By formulae

If θ is measured in radian measure, then

1) $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, 2) $\lim_{\theta\to 0}\cos\theta=1$, 3) $\lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1$.

Examples:

1) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ **[B.T.E.2015]** 2) $\lim_{x\to 0}\frac{\sin 4x-\sin 6x}{x}$ **[IoPE 2014]**

3) $\lim_{x\to 0}\frac{\cos 5x-\cos 3x}{x^2}$ **[B.T.E.2016]** 4) $\lim_{x\to 0}\frac{3\sin 2x+2x}{3x+2\tan 3x}$ **[IoPE 2013]**

Examples for Tutorial

Evaluate the following limits:

1) $\lim_{x\to 0} x\operatorname{cosec} x$ **[B.T.E.2017] [1]**

2) $\lim_{x\to 0} \frac{\sin 3x}{\tan 5x}$ **[B.T.E.2015] [3/5]**

3) $\lim_{x\to 0}\left(\frac{3\sin x+4x}{7x-2\tan x}\right)$ **[B.T.E.2016] [7/5]**

4) $\lim_{x\to 0} \frac{\sin 2x-2\sin x}{x^3}$ **[B.T.E.2015] [-1]**

5) $\lim_{x\to 0} \frac{2\sin x^0-\sin 2x^0}{x^0}$ $\left[(\pi/180)^3\right]$

Exponential and Logarithmic Limits:

Type I

1) $\lim_{x\to 0}(1+x)^{1/x}=e$ where $e\approx 2.718$

Note: $\lim_{x\to 0}(1+kx)^{1/x}=e^k$.

Examples

1) $\lim_{x\to 0}(1+2x)^{4/x}$ **[IoPE 2012]**

2) $\lim_{x\to 0}\left(\frac{3+2x}{3-2x}\right)^{1/x}$ **[IoPE 2015]**

3) $\lim_{x\to 0}\left(1+\sin x\right)^{2\operatorname{cosec} x}$

Examples for Tutorial

1) $\lim_{x\to 0}\left(\frac{1+x}{1-x}\right)^{1/x}$ **[B.T.E.2016]** $\left[e^2\right]$

2) $\lim_{x\to 0}\left(\frac{3+2x}{3-5x}\right)^{1/x}$ $[e^{7/3}]$

3) $\lim_{x\to 0}(x)^{\frac{1}{x-1}}$

4) $\lim_{x\to \pi/2}\left(1+\cos x\right)^{3\sec x}$ $\left[e^3\right]$

5) $\lim_{x\to 0}\left(1+\tan x\right)^{3\cot x}$ $\left[e^3\right]$

Type II

$\lim_{x\to\infty}(1+1/x)^x=e$, where $e\approx 2.718$.

Note: $\lim_{x\to\infty}(1+k/x)^x=e^k$

Examples

1) $\lim_{x\to\infty}\left(\frac{x+3}{x+2}\right)^x$ **[IoPE 2015]**

2) $\lim_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^x$ **[IoPE 2012]**

3) $\lim_{x\to\infty}\left(1+e^{-x}\right)^{e^x}$

Examples for Tutorial

1) $\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x$ **[B.T.E.2015]** $[1/e]$ 2) $\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^{x+2}$ **[B.T.E.'91]** $[e^2]$

3) $\lim_{x\to\infty}\left(\frac{2x+3}{2x+1}\right)^{x+5}$ $[e]$ 4) $\lim_{n\to\infty}\left[1+\frac{1}{n+1}\right]^n$ **[B.T.E.'86]** $[e]$

Type III

1) $\lim_{x\to 0}\frac{a^x-1}{x}=\log a$ 2) $\lim_{x\to 0}\frac{\log(1+mx)}{x}=m$

Examples

1) $\lim_{x\to 0}\frac{5^x-4^x}{\tan 2x}$ **[IoPE2010]** 2) $\lim_{x\to 0}\frac{3^x+3^{-x}-2}{x^2}$ **[B.T.E.15]**

3) $\lim_{x\to 0}\frac{15^x-5^x-3^x+1}{x\sin x}$ **[B.T.E.2017]** 4) $\lim_{x\to 0}\frac{(5^x-1)\tan x}{\sqrt{x^2+16}-4}$ **[B.T.E.2015,16]**

Examples for Tutorial

1) $\lim_{x\to 0}\frac{a^x+b^x-2}{x}$ **[B.T.E.2017]** $\left[(\log a)^2\right]$ 2) $\lim_{x\to 0}\left(\frac{6^x-3^x-2^x+1}{x^2}\right)$ **[B.T.E.2015,16]** $\left[\log 3.\log 2\right]$

3) $\lim_{x\to 0}\frac{10^x-2^x-5^x+1}{x^2}$ **[B.T.E.2016]** $\left[\log 5.\log 2\right]$ 4) $\lim_{x\to 0}\frac{3^{2x}-2^{3x}}{\sin x}$ **[B.T.E.2015]** $\left[\log(9/8)\right]$

5) $\lim_{x\to 0}\frac{\log(1+5x)}{x}$ **[B.T.E.2017]** **[5]**

Type IV

$\lim_{x\to 0}\frac{e^x-1}{x}=\log_e e=1$

Note: $\lim_{x\to 0}\frac{e^{kx}-1}{x}=k$

Examples

1) $\lim_{x\to 0}\frac{\tan x}{e^{2x}-e^x}$ **[IoPE 2015]** 2) $\lim_{x\to 0}\frac{e^{2x}-1}{e^x-1}$ 3) $\lim_{x\to 0}\frac{e^{2x}+e^{-x}+e^{3x}-3}{x}$ **[IoPE 2013]**

Examples for Tutorial

1) $\lim_{x\to 0}\frac{e^{2x}-1}{x}$ **[2]** 2) $\lim_{x\to 0}\frac{e^x-e^{-x}}{x}$ **[B.T.E.'92]** **[2]**

3) $\lim_{x\to 0}\frac{e^{2x}-e^x}{\sin x}$ **[1]** 4) $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x^2}$ **[1]** 5) $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos 2x}$ **[1/2]**