# Laplace Transform

**Definition of Laplace Transform [IoPE2010]**

Let f(t) be a function of t defined for all positive values of t, then the Laplace Transform of f(t) is defined by

$$L[f(t)] = f(s) = \int_0^\infty e^{-st} f(t)dt$$

provided the integral exists. Here s is a parameter which may be real or complex number.

**Laplace Transform of Some Elementary Functions**

(1) $L(1) = \frac{1}{s}, s \neq 0$ (2) $L(e^{at}) = \frac{1}{s-a}, \; s > a$

(3) $L(\sin at) = \frac{a}{s^2 + a^2}, \; s > 0$ (4) $L(\cos at) = \frac{s}{s^2 + a^2}, \; s > 0$

(5) $L(\sinh at) = \frac{a}{s^2 - a^2}, \; s > |a|$ (6) $L(\cosh at) = \frac{s}{s^2 - a^2}, \; s > |a|$

(7) $L(t^n) = \frac{n!}{s^{n+1}}$ $where\ n = 0,1,2,3,---$

**Linearity Property of Laplace Transform [IoPE 2013]**

If f,g,h are functions of t and a,b,c are constants, then

$$L[a\,f(t) + b\,g(t) + ch(t)] = a\,L[f(t)] + b\,L[g(t)] + c\,L[h(t)]..$$

## Examples

Find the Laplace Transforms of the following:

1) $4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t + 2$ **[IoPE2017]** 2) $\sin 5t + \sin^3 t$ **[IoPE 2013]** 3) $e^{at}$ **[IoPE 2009]**

## Examples for Tutorial

Find the Laplace Transform of the following:

1) $A + Bt + Ct^2$ **[IoPE2014]** $[\frac{A}{s} + \frac{B}{s^2} + \frac{2C}{s^3}]$

2) $3\ e^{2t} - 2\sin 3t$ **[IoPE 2009]** $[\frac{3}{s-2} - \frac{6}{s^2+9}]$

3) sin(at + b) **[IoPE 2012]** $[\cos b\left(\frac{a}{s^2+a^2}\right) + \sin b\left(\frac{s}{s^2+a^2}\right)]$

4) 4 sin4t + cos4t +4t **[IoPE 2007]** $[\frac{16}{s^2+16} + \frac{s}{s^2+4} + \frac{4}{s^2}]$

5) sin2t sin5t **[IoPE 2013]** $\left[\frac{20s}{(s^2+9)(s^2+49)}\right]$

6) $\sin^2 3t$ **[IoPE 2008]** $[\frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]]$

7) $\sin^3 3t$ **[IoPE 2014]** $[\frac{162}{(s^2+9)(s^2+81)}]$

8) $3\cosh 5t - 4\sinh 5t$ $\left[\frac{3s-20}{(s^2-25)}\right]$

## First Shifting Property

If $L[F(t)] = \overline{f}(s)$, then $L[e^{at}F(t)] = \overline{f}(s-a)$.

Following useful formulae are the application of this property.

(1) $L(e^{at}t^n) = \dfrac{n!}{(s-a)^{n+1}}, s > a$ (2) $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2 + b^2}, s > a$

(3) $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}, s > a$ (4) $L(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2 - b^2}, s > a$

(5) $L(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}, s > a$

**Note:** Hyperbolic sine and cosine functions are given as follows:

1) $\sinh x = \dfrac{e^x - e^{-x}}{2}$, 2) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

# Examples

Find the Laplace transform of the following functions:

1) $e^{2t}t^2$ 2) $e^{-2t}(2\cos 5t - 4\sin 5t)$ **[IoPE 2012]**

## Examples for Tutorial

Find by using the First Shifting Theorem:

1) $L(e^{2t}t^3)$ $\left[\dfrac{6}{(s-2)^4}\right]$

2) $L[e^{-3t}(\cos 4t + 3\sin 4t)]$ **[IoPE2014]** $\left[\dfrac{s+15}{s^2+6s+25}\right]$

3) $L(e^{-t}\cosh 2t)$ **[IoPE2010]** $\left[\dfrac{s+1}{s^2+2s-3}\right]$

4) $L(\sin 3t \sinh 2t)$ $\left[\dfrac{12s}{s^4+10s^2+169}\right]$

## Second Shifting Property

If $L[F(t)] = f(s)$ and $G(t) = \begin{cases} 0 & \text{when } t < a \\ F(t-a) & \text{when } t > a \end{cases}$ then $L[F(t)] = e^{-as} f(s)$.

## Definition of Inverse Laplace Transform

If L[f(t)] = f(s), then f(t) is called the inverse Laplace Transform of f(s) and it is denoted by $f(t) = L^{-1}[f(s)]$, where $L^{-1}$ is inverse Laplace Transform operator.

## Linearity Property of Inverse Laplace Transform

If $L^{-1}[f(s)] = f(t)$ and $L^{-1}[g(s)] = g(t)$ then

$$L^{-1}[a\,f(s) + b\,g(s)] = aL^{-1}[f(s)] + bL^{-1}[g(s)]$$

where a and b are constants.

## Formulae on Inverse Laplace Transform

1) $L^{-1}(\frac{1}{s}) = 1, s \neq 0$  2) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}, \ s \neq a$  3) $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

4) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$  5) $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$  6) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

7) $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$

## Formulae of inverse Laplace Transform using First Shifting Theorem:

1) $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{e^{at}\sin bt}{b}, s > a$  2) $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt, \ s > a$

3) $L^{-1}\left[\frac{1}{(s-a)^2-b^2}\right] = \frac{e^{at}\sinh bt}{b}, s > a$  4) $L^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right] = e^{at}\cosh bt, \ s > a$

5) $L^{-1}\left[\frac{1}{(s-a)^{n+1}}\right] = \frac{e^{at}t^n}{n!}, s > a$

## Inverse Laplace Transforms – Method of Partial Fractions:

We have seen that the Laplace transform of a function is a rational algebraic function. So, we express the given function into partial fractions and then to find the inverse Laplace transform by standard formulae.

**Examples:** Find

1) $L^{-1}\left(\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right)$ 2) $L^{-1}\left(\frac{3s+2}{s^2-s-2}\right)$ 3) $L^{-1}\left[\frac{4s+5}{(s+2)(s-1)^2}\right]$ 4) $L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right]$

## Examples for Tutorial

Find the inverse Laplace Transform of the following functions:

1) $L^{-1}\left[\frac{1}{s(s+4)}\right]$. **[IoPE 2006]**  $\left[\frac{1}{4} - \frac{1}{4}e^{-4t}\right]$

2) $L^{-1}\left[\frac{1-7s}{(s-3)(s-1)(s+2)}\right]$. **[IoPE2016]**  $\left[-2e^{3t} + e^{t} + e^{-2t}\right]$

3) $L^{-1}\left(\frac{3s}{s^2+2s-8}\right)$ **[IoPE 2016]**  $\left[2e^{-4t} + e^{2t}\right]$

4) $L^{-1}\left[\frac{2s+1}{s^3-4s}\right]$  $[-\frac{1}{4} - \frac{3}{8}e^{-2t} + \frac{5}{8}e^{2t}]$

5) $L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right]$  $[\frac{1}{2}(e^{t} - \cos t - \sin t)]$

## Laplace Transform of Derivatives

If $L[F(t)] = f(s)$ then

$$\mathrm{L}\left[\frac{d^n}{dt^n}F(t)\right] = s^n f(s) - s^{n\text{-}1}F(0) - s^{n\text{-}2}F'(0) - - - sF^{n\text{-}2}(0) - F^{n\text{-}1}(0).$$

**Note: 1)** Laplace transform of first order derivative is $L[F'(t)] = s\,f(s) - f(0).$

2) $L(x) = \bar{x}$ 3) $L(y) = \bar{y}$

4) $L\left[\frac{dy}{dt}\right] = L(y') = s\,\bar{y} - y(0)$ where $y(0)$ is the of $y$ at $t = 0.$

5) $L\left[\frac{dx}{dt}\right] = L(x') = s\,\bar{x} - x(0)$ where $x(0)$ is the of $x$ at $t = 0.$

## Solution of differential equations using Laplace transform

The Laplace transform can be used to solve differential equations of first order and first degree.

Steps to solve examples:

1) Take the Laplace transform on both sides of the given differential equation and apply initial conditions.

2) Collect the terms with $\bar{x}$ or $\bar{y}$ on L.H.S. and remaining terms on R.H.S.

3) Then find $\bar{x}$ or $\bar{y}$ as a function of s .

4) Resolve f(s) into partial fraction.

5) Then take inverse Laplace transform on both sides. This gives $x$ or $y$ as a function of t.

## Examples

Solve the differential equations using Laplace transform using initial conditions.

1) $\frac{dy}{dt} - y = 3e^{-2t}, \; y(0) = -1$

2) $3x' + 2x = e^{3t}, \quad x(0) = 1$

## Examples for Tutorial

1) $\frac{dx}{dt} + x = 3, x(0) = 2$ $[x(t) = 3 - e^{-t}]$

2) $y' + y = \sin t$ , $y = 0$ when t = 0. $\left[y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)\right]$