Integration

1) The symbol $\int$ stands for integral.

2) **Integration of a function:** If $\frac{d[f(x)]}{dx} = g(x)$**,** then g(x) is called integral of f(x) and it is denoted by $\int g(x)dx = f(x) + c$, where 'c' is called an arbitrary constant.

3) Integration is also called as anti-derivative or primitive.

4) Integration is reverse process of differentiation.

Rules of Integration:

1) If k is any constant , then $\int k.f(x)\,dx = k\int f(x)\,dx$.

2) If $f(x)$ and $g(x)$ are two functions of x, then $\int [.f(x) \pm g(x)]dx = \int f(x)\,dx \pm \int g(x)\,dx$.

Standard Substitutions:

Integrand	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$ or $x = a\sin 2\theta$
$\sqrt{\frac{a-x}{x}}$	$x = a\sin^2\theta$ or $x = a\cos^2\theta$
$\sqrt{\frac{a+x}{x}}$	$x = a\tan^2\theta$ or $x = a\cot^2\theta$
$\sqrt{(x-a)(b-x)}$ or $\sqrt{\frac{x-a}{b-x}}$	$x = a\sin^2\theta + b\cos^2\theta$

Integration of standard functions (Formulae) -

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ where $n \neq -1$, 2) $\int dx = x + c$,

3) $\int \frac{dx}{x} = \log x + c$ where $x > 0$, 4) $\int a^x dx = \frac{a^x}{\log a} + c$ where $a > 0$ and $a \neq 1$, 5) $\int e^x dx = e^x + c$,

6) $\int \sin x dx = -\cos x + c$, 7) $\int \cos x dx = \sin x + c$, 8) $\int \sec^2 x dx = \tan x + c$,

9) $\int \operatorname{cosec}^2 x dx = -\cot x + c$, 10) $\int \sec x \tan x dx = \sec x + c$, 11) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$,

12) $\int \tan x dx = \log|\sec x| + c$, 13) $\int \cot x dx = \log|\sin x| + c$,

14) $\int \sec x dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + c$,

15) $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + c = \log\left|\tan\left(\frac{x}{2}\right)\right| + c$,

16) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c = -\cos^{-1} x + c$, 17) $\int \frac{dx}{1+x^2} = \tan^{-1} x + c = -\cot^{-1} x + c$,

18) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c = -\operatorname{cosec}^{-1} x + c$,

19) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + c = -\cos^{-1}(x/a) + c$,

20) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a) + c = -\frac{1}{a}\cot^{-1}(x/a) + c$,

21) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}(x/a) + c = -\frac{1}{a}\operatorname{cosec}^{-1}(x/a) + c$,

22) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x + \sqrt{x^2+a^2}\right) + c$, 23) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x + \sqrt{x^2-a^2}\right) + c$,

24) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$, 25) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$,

26) $\int uv dx = u\int v dx - \int \frac{du}{dx}\left[\int v dx\right] dx$

27) $\int \sqrt{a^2-x^2} dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + c$,

28) $\int \sqrt{x^2-a^2} dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x + \sqrt{x^2-a^2}) + c$,

29) $\int \sqrt{x^2+a^2} dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x + \sqrt{x^2+a^2}) + c$,

30) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + c$, 31) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + c$

32) $\int e^x\left[f(x) + f'(x)\right] dx = e^x f(x) + c$

Note - $f(x) = \int f'(x) dx + c$

Integration of Algebraic Functions

Now, we solve integration of some algebraic functions by using standard formulae.

Examples:

1) $\int [x^2 + e^{\log x} + (e/2)^x]\, dx$ **[IoPE 2016]**

2) $\int \left(\frac{1}{49 - 4x^2} - \frac{1}{x\sqrt{x^2 - 25}} \right) dx.$ **[IoPE 2006]**

3) $\int \left[e^{2-5x} + \frac{2}{6x+1} \right] dx$ **[IoPE 2014]**

Examples for Tutorial

1) $\int (3x+4)^2\, dx$ **[IoPE 2007]** $\quad [I = \frac{x^4}{4} + \frac{3^x}{\log 3} + 27x + e^x + \frac{x^{e+1}}{e+1} + c]$

2) $\int \frac{1 + x - x^2}{\sqrt{x}}\, dx$. **[B.T.E.'79,IoPE 2009]** $\quad \left[I = 2x^{1/2} + \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + c \right]$

3) $\int \left(\frac{x}{m} + \frac{m}{x} + x^m + m^x \right) dx$.**[IoPE 2012, B.T.E.'88]** $\quad [I = \frac{1}{m} \cdot \frac{x^2}{2} + m \log x + \frac{x^{m+1}}{m+1} + \frac{m^x}{\log m}]$

4) $\int \left(\frac{1}{\sqrt{4 - 9x^2}} + \frac{1}{2x^2 + 3} \right) dx$.**[IoPE 2013]** $\quad [I = \frac{1}{3}\sin^{-1}(3x/2) + \frac{1}{\sqrt{6}}\tan^{-1}(\sqrt{2}x\sqrt{3}/) + c]$

Integration of trigonometric functions

Now, we solve integration of some trigonometric functions with the help of trigonometric formulae.

Examples

1) $\int \frac{dx}{\cos^2 x \sin^2 x}$. **[IoPE 2016]**

2) $\int \sqrt{1 + \sin 2x}\, dx$ **[IoPE2007, B.T.E.',92]**

3) $\int (\tan x - \cot x)^2 dx$ **[IoPE 2015]**

4) $\int \frac{1 - \cos x}{1 + \cos x} dx$. **[IoPE2014]**

Examples for Tutorial

1) $\int \sin^2 x \cos^2 x\, dx$.**[IoPE2008]** $\quad [I = \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right] + c]$

2) $\int (\sec^2 x + \text{cosec}^2 x) dx$ **[B.T.E.'79]** $\quad [I = \tan x - \cot x + c]$

3) $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx$ $\quad [I = -\cot x - \tan x + c]$

4) $\int \sin 5x \cos 3x\, dx.$ **[IoPE 2007]** $\quad \left[I = -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + c \right]$

Examples on the formula : $f(x)=\int f'(x)dx+c$

1) If $f'(x)=6x^2+ax+7$, $f(0)=2$ and $f(1)=6$, find f(x). **[IoPE 2007]**

2) If $f'(x)=\frac{1}{x}+\frac{1}{1+x^2}$ and $f(1)=\frac{\pi}{4}$, find f(x). **[IoPE 2016,B.T.E.'83]**

Examples for Tutorial

1) If $f'(x)=ax+b$ and $f(1)=5$, $f(2)=13$ and $f(0)=1$, then find $f(x)$. $[f(x)=2x^2+2x+1]$

2) If $f'(x)=7+6x-3x^2$ and $f(-1)=0$, then find $f(x)$. $[f(x)=3+7x+3x^2-x^3]$

Integration by Substitution:

Sometimes it is difficult to solve $\int f(x)dx$. But, by changing the variable of integration by a suitable substitution, the function f(x) can be reduced to standard form and it will be solved easily. So, this method is called method of substitution.

Examples

1) $\int \frac{x+1}{x^2+2x+7}dx$ 2) $\int \sec^4 x \tan^3 x\, dx$. **[IoPE 2015]** 3) $\int \sqrt{1+e^x}e^x\,dx$ **[IoPE 2016]**

Examples for Tutorial

1) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ **[IoPE 2008]** $[I=-2\cos\sqrt{x}+c]$

2) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$. **[IoPE 2007]** $[I=2\sin\sqrt{x}+c]$

3) $\int x^3\sin x^4\,dx$ $[I=-\frac{1}{4}(x^4+1)\cos(x^4)]$

4) $\int \frac{\sin(\tan^{-1}x)}{1+x^2}dx$ **[IoPE 2016]** $[I=-\cos(\tan^{-1}x)+c]$

Integral of the type: $\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$.

We use following method to solve above integrals.

a) Take the coefficient of x^2 (i.e. a) outside the integral so that the coefficient of x^2 should be always ± 1.

b) Then complete the perfect square in the denominator by using the formula:

$$\text{Third Term}=\left[\frac{1}{2}\left(\text{Coefficient of } x\right)\right]^2$$

c) Then the given integral reduced to standard form and can be solved easily.

Examples

1) $\int \frac{dx}{\sqrt{1-4x-x^2}}$ **[IoPE2009]**

2) $\int \frac{1}{3+2x-x^2}\,dx$ **.[IoPE2015]**

Examples for Tutorial

1) $\int \frac{dx}{\sqrt{x^2-4x-5}}$ **[IoPE2016]** $\left\{I = \log\left[(x-2)+\sqrt{x^2-4x-5}\right]+c\right\}$

2) $\int \frac{dx}{x^2-4x+29}$ **[IoPE2011]** $\left[I = \frac{1}{5}\tan^{-1}[\frac{x-2}{5}]+c\right]$

3) $\int \frac{1}{x^2+6x+25}\,dx$ **[B.T.E.'89]** $\left[I = \frac{1}{4}\tan^{-1}\left(\frac{x+3}{4}\right)\right]$

Integration of Rational Functions

The expression of the form $\frac{P(x)}{Q(x)}$ is called rational function. We express this function as a sum or difference of functions by suitable methods as follows:

1) If the degree of numerator is equal to or greater than the degree of denominator, then divide numerator by denominator.

2) If the degree of numerator and denominator are equal, then we adjust denominator in the numerator.

Examples:

1) $\int \frac{2x+1}{3x-2}\,dx$

2) $\int \frac{x^2-1}{x^2+1}\,dx$ **[B.T.E.'87]**

Examples for Tutorial

1) $\int \frac{2x+1}{x+2}\,dx$ **[B.T.E.'87]** $\left[I = 2x-3\log(x+2)+c\right]$

2) $\int \frac{x^2-3x+4}{2x+1}\,dx$ $\left[I = \frac{x^2}{4}-\frac{7}{4}x+\frac{23}{8}\log(2x+1)+c\right]$

3) $\int \frac{2x^2+5x+1}{3x+2}\,dx.$ **[IoPE 2009]** $\left[I = \frac{x^2}{3}+\frac{11}{9}x-\frac{13}{27}\log(3x+2)+c\right]$

Integrals of the type: $\int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a\sin^2 x+b\cos^2 x}$ and $\int \frac{dx}{a\sin^2 x+b\cos^2 x+c}$.

We use following method to solve above integrals.

a) Divide numerator and denominator by $\cos^2 x$.

b) Then we use the trigonometric formula $1+\tan^2 x=\sec^2 x$ in the denominator only if necessary.

c) Put tanx = t in the denominator only. $\therefore \sec^2 x\,dx = dt$

d) Then the given integral reduced to standard form and can be solved easily.

Examples

1) $\int \frac{1}{3-2\sin^2 x}dx$. **[IoPE 2006]** 2) $\int \frac{dx}{\sin^2 x+2\cos^2 x+3}$. **[IoPE 2015]**

Examples for Tutorial

1) $\int \frac{1}{3+2\sin^2 x}dx$ $\left[I=\frac{1}{\sqrt{15}}\tan^{-1}(\sqrt{\frac{5}{3}}\tan x)+c\right]$

2) $\int \frac{1}{2+3\cos^2 x}dx$ $\left[I=\frac{1}{\sqrt{10}}\tan^{-1}\left(\sqrt{\frac{2}{5}}\tan x\right)+c\right]$

3) $\int \frac{dx}{3\sin^2 x+4\cos^2 x}$. **[IoPE 2009]** $[I=\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{2}\tan x\right)+c]$

Integrals of the type: $\int \frac{dx}{a+b\cos x}, \int \frac{dx}{a+b\sin x}, \int \frac{dx}{a\sin x+b\cos x}$ and $\int \frac{dx}{a\sin x+b\cos x+c}$.

We use following method to solve above integrals.

a) We use the substitution $\tan(x/2)=t$. Then,

$$\sin x=\frac{2\tan(x/2)}{1+\tan^2(x/2)}=\frac{2t}{1+t^2}, \cos x=\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}=\frac{1-t^2}{1+t^2} \text{ and } dx=\frac{2dt}{1+t^2}.$$

b) Then the given integrals reduced to standard forms and can be solved easily.

Examples:

1) $\int \frac{dx}{2\sin x+3\cos x}$ **[IoPE 2013]** **2)** $\int \frac{dx}{1+\sin x+\cos x}$. **[IoPE 2010, B.T.E.'91]**

Examples for Tutorial

1) $\int \frac{dx}{3-2\sin x}$. **[IoPE 2008, B.T.E.'92]** $[I=\frac{2}{\sqrt{5}}\tan^{-1}\left[\frac{3\tan\left(\frac{x}{2}\right)-2}{\sqrt{5}}\right]+c\ ]$

2) $\int \frac{dx}{\sin x+\cos x}$ **[IoPE 2009]** $[I=\frac{1}{\sqrt{2}}\log\left[\frac{\tan\left(\frac{x}{2}\right)-1+\sqrt{2}}{\tan\left(\frac{x}{2}\right)-1-\sqrt{2}}\right]+c\ ]$

3) $\int \frac{dx}{2+\cos x-\sin x}$ **[IoPE 2011]** $\left[I=\sqrt{2}\tan^{-1}[(\tan(x/2)-1)/\sqrt{2}]+c\right]$

Integration by Parts

This method is generally useful whenever there is integration of product of two functions.

Statement: If u and v are functions of x, then $\int uvdx=u\int vdx-\int\left(\frac{du}{dx}\int vdx\right)dx.$

Note: 1) The selection of the first function u is that function which comes first in the serial order of the letters "**LIATE**" where

L – Logarithmic function

I – Inverse trigonometric function

A – Algebraic function

T – Trigonometric function

E – Exponenial function

and the remaining function is the second function v.

2) Generally the second function v is that function whose integral is known.

3) If only-one function is given, then that function is u function and the remaining function is v = 1.

Examples

1) $\int x\sin^2 x\,dx$

2) $\int x^2 \cos x\,dx$. **[IoPE 2006]**

3) Prove that $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} - \frac{a^2}{2}\sin^{-1}(x/a)+c$ **[[IoPE 2013]**

Examples for Tutorial

1) $\int x\sec^2 x\,dx$ $[I = x\tan x - \log(\sec x)+c]$

2) $\int x^2 e^{3x}\,dx$ **[[IoPE .2008]** $[I = \left(x^2 - \frac{2}{3}x + \frac{2}{9}\right)\frac{e^{3x}}{3} + c]$

3) $\int (\log x)^2 dx$ **[IoPE 2009]** $[I = x(\log x)^2 - 2\,x\,\log x + 2x + c$

4) $\int x^2 \log x\,dx$ **[[IoPE 2010]** $[\mathrm{I} = \frac{x^3}{3}\left(\log x - \frac{1}{3}\right) + c]$

Prove that

1) $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2})+c$ **[IoPE 2014]**

2) $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2-a^2})+c$ **[IoPE 2014]**

3) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]+c$ **[IoPE 2006]**

4) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]+c$ **[IoPE 2006]**

Examples on a special integral - $\int e^x [f(x) + f'(x)]dx = e^x f(x) + c$

1) $\int e^x \frac{(x+2)}{(x+3)^2}dx$ **[IoPE 2014, B.T.E.'88]**

2) $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$ **[IoPE 2015]**

3) $\int [\tan(\log x) + \sec^2(\log x)]dx$ **[IoPE 2011]**

Examples for Tutorial

1) $\int e^x \frac{x}{(x+1)^2}dx$ **[B.T.E.'91]** $\left[I = e^x \frac{1}{x+1} + c\right]$

2) $\int e^x(\sin x + \cos x)\,dx$ [**B.T.E.'86]** $[I = e^x \sin x + c]$

3) $\int e^x \frac{(\cos x - \sin x)}{\sin^2 x}dx$ **[IoPE 2014]** $\left[I = -e^x \operatorname{cosec} x + c\right]$

4) $\int [\sin(\log x) + \cos(\log x)]dx$ **[IoPE 2013, B.T.E.'92]** $[I = x\sin(\log x) + c\,]$