Important Concept and Formulae

Algebraic Identities:

1) $(a+b)^2 = a^2 + 2ab + b^2$, 2) $(a-b)^2 = a^2 - 2ab + b^2$, 3) $a^2 - b^2 = (a+b)(a-b)$,

4) $(x+a)(x+b) = x^2 + (a+b)x + ab$, 5) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$,

6) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + 3ab(a+b) + b^3$, 7) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - 3ab(a-b) - b^3$,

8) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, 9) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$,

10) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - 2ab - 2bc - 2ca)$, 11) $a^4 - b^4 = (a-b)(a+b)(a^2 + b^2)$.

Laws of Logarithms:

1) If $y = \log_a x, a > 0, a \neq 1, x > 0$, then $x = a^y$.

2) $\log_a(mn) = \log_a m + \log_a n$, $m, n, a > 0, a \neq 1$, 3) $\log_a(m/n) = \log_a m - \log_a n$, $m, n, a > 0, a \neq 1$,

4) $\log_a m^n = n \log_a m$, $m, n, a > 0, a \neq 1$, 5) $\log_n m = \frac{\log_a m}{\log_a n}$, $m, n, a > 0, a \neq 1$,

6) $\log_n m = \frac{1}{\log_m n}$, $m, n > 0, m \neq 1, n \neq 1$, 7) $\log_a 1 = 0, a > 0, a \neq 1$, 8) $\log_a a = 1, a > 0, a \neq 1$,

9) $a^{\log_a x} = x$, $a, x > 0, a \neq 1$, 10) $e^{\log_e x} = x$, $e, x > 0$, $e \approx 2.718$.

Laws of Indices:

1) $a^m a^n = a^{m+n}$, 2) $(a^m)^n = a^{mn}$, 3) $(a \times b)^m = a^m \times b^m$, 4) $\frac{a^m}{a^n} = a^{m-n}$,

5) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, 6) $a^{-m} = \frac{1}{a^m}$, 7) $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$, 8) $a^0 = 1$.

Trigonometry

1) The unit of angle measure is degree and it is denoted by ^{0}.

2) The unit of angle measure is radian and it is denoted by ^{c}.

3) π radian = 180 degree i.e. $\pi^C = 180^0$, 1 radian = $\left(\frac{180}{\pi}\right)$ degree i.e. $1^C = \left(\frac{180}{\pi}\right)^0$,

θ radian = $\left(\theta \times \frac{180}{\pi}\right)$ degree i.e. $\theta^C = \left(\theta \times \frac{180}{\pi}\right)^0$

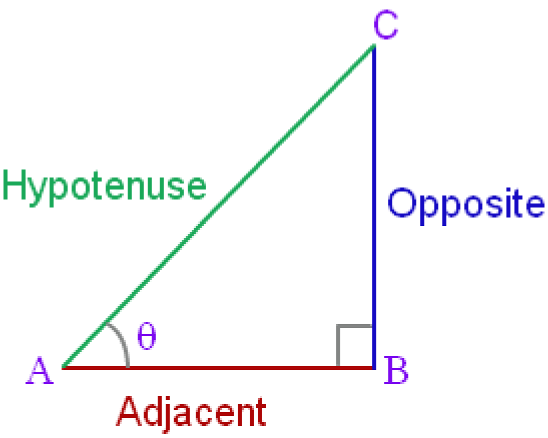
4) 1 degree = $\left(\frac{\pi}{180}\right)$ radian i.e. $1^0 = \left(\frac{\pi}{180}\right)^C$, x degree = $\left(x \times \frac{\pi}{180}\right)$ radian i.e. $x^0 = \left(x \times \frac{\pi}{180}\right)^C$

5) $1^C = 57^0 17' 48'' = 57.3^0$ (approx.), $1^0 = 0.01745^c$ (approx.)

6) 1 degree = 60 minutes i.e. $1^0 = 60'$, 1 minute = 60 seconds i.e. $1' = 60''$.

Trigonometric Ratios:

ΔABC is a right angled triangle as shown in the figure. AB is adjacent side, BC is opposite side and AC is hypotenuse, then



1) $\sin\theta = \dfrac{\text{Opposite side}}{\text{Hypotenuse}} = \dfrac{BC}{AC}$, 2) $\cos\theta = \dfrac{\text{Adjacent side}}{\text{Hypotenuse}} = \dfrac{AB}{AC}$, 3) $\tan\theta = \dfrac{\text{Opposite side}}{\text{Adjacent side}} = \dfrac{BC}{AB}$,

4) $\text{cosec}\,\theta = \dfrac{\text{Hypotenuse}}{\text{Opposite side}} = \dfrac{AC}{BC}$, 5) $\sec\theta = \dfrac{\text{Hypotenuse}}{\text{Adjacent side}} = \dfrac{AC}{AB}$, 6) $\cot\theta = \dfrac{\text{Adjacent side}}{\text{Opposite side}} = \dfrac{AB}{BC}$.

Inter-relation between trigonometric ratios:

1) $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$, 2) $\sec\theta = \dfrac{1}{\cos\theta}$, 3) $\tan\theta = \dfrac{1}{\cot\theta}$, 4) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, 5) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

6) $\sin^2\theta + \cos^2\theta = 1$, $\sin^2\theta = 1 - \cos^2\theta$, $\cos^2\theta = 1 - \sin^2\theta$

7) $1 + \tan^2\theta = \sec^2\theta$, $\sec^2\theta - \tan^2\theta = 1$, $\tan^2\theta = \sec^2\theta - 1$

8) $1 + \cot^2\theta = co\sec^2\theta$, $co\sec^2\theta - \cot^2\theta = 1$, $\cot^2\theta = co\sec^2\theta - 1$

Trigonometric ratios of negative angles:

1) $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$,

2) $\text{cosec}(-\theta) = -\text{cosec}\,\theta$, $\sec(-\theta) = \sec\theta$, $\cot(-\theta) = -\cot\theta$.

Trigonometric functions of standard angles:

Angle / Ratio	0^0	$30^0 = (\pi/6)^c$	$45^0 = (\pi/4)^c$	$60^0 = (\pi/3)^c$	$90^0 = (\pi/2)^c$	$180^0 = \pi$	$270^0 = (3\pi/2)^c$	$360^0 = (2\pi)^c$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	-1	∞
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1	∞	1
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞

Signs of trigonometric functions:

Trigonometric ratios \ Quadrants	I	I I	III	IV
sin	+	+	-	-
cos	+	-	-	+
tan	+	-	+	-

Trigonometric functions of addition and subtraction:

1) $\sin(A+B)=\sin A\cos B+\sin B\cos A,$ 2) $\sin(A-B)=\sin A\cos B-\sin B\cos A,$

3) $\cos(A+B)=\cos A\cos B-\sin A\sin B,$ 4) $\cos(A-B)=\cos A\cos B+\sin A\sin B,$

5) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A.\tan B},$ 6) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A.\tan B},$

7) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B,$ 8) $\cos(A+B)\cos(A-B)=\cos^2 A-\cos^2 B.$

Trigonometric functions of allied angles

Ratio \ Angle	$-\theta$	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$	$2\pi+\theta$
sin	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
cos	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
cosec	$-\text{cosec}\,\theta$	$\sec\theta$	$\sec\theta$	$\text{cosec}\,\theta$	$-\text{cosec}\,\theta$	$-\sec\theta$	$-\sec\theta$	$-\text{cosec}\,\theta$	$\text{cosec}\,\theta$
sec	$\sec\theta$	$\text{cosec}\,\theta$	$-\text{cosec}\,\theta$	$-\sec\theta$	$-\sec\theta$	$-\text{cosec}\,\theta$	$\text{cosec}\,\theta$	$\sec\theta$	$\sec\theta$
cot	$-\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\tan\theta$

Trigonometric ratios of large angles:

1) When n is an even integer then,

$a)\ \sin\left(n\frac{\pi}{2}\pm\theta\right)=\pm\sin\theta,\quad b)\ \cos\left(n\frac{\pi}{2}\pm\theta\right)=\pm\cos\theta,\quad c)\ \tan\left(n\frac{\pi}{2}\pm\theta\right)=\pm\tan\theta,$ etc.

2) When n is an odd integer then,

$a)\ \sin\left(n\frac{\pi}{2}\pm\theta\right)=\pm\cos\theta,\quad b)\ \cos\left(n\frac{\pi}{2}\pm\theta\right)=\pm\sin\theta,\quad c)\ \tan\left(n\frac{\pi}{2}\pm\theta\right)=\pm\cot\theta,$ etc.

The correct sign is to be chosen depending the quadrant in which the angle lies, using the sign convention.

Trigonometric functions of double and triple angles:

1) $\sin 2\theta=2\sin\theta\cos\theta,$ 2) $\cos 2\theta=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1,$

3) $\sin 2\theta=\dfrac{2\tan\theta}{1+\tan^2\theta},$ 4) $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta},$ 5) $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

6) $1-\cos 2\theta=2\sin^2\theta,\quad \therefore \sin^2\theta=\dfrac{1-\cos 2\theta}{2},$ 7) $1+\cos 2\theta=2\cos^2\theta,\quad \therefore \cos^2\theta=\dfrac{1+\cos 2\theta}{2},$

8) $1+\sin 2\theta=(\cos\theta+\sin\theta)^2,$ 9) $1-\sin 2\theta=(\cos\theta-\sin\theta)^2,$

10) $\sin 3\theta=3\sin\theta-4\sin^3\theta,$ 11) $\cos 3\theta=4\cos^3\theta-3\cos\theta,$ 12) $\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^3\theta}.$

Trigonometric functions of half angles:

1) $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$, 2) $\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = 1 - 2\sin^2(\theta/2) = 2\cos^2(\theta/2) - 1$,

3) $\sin\theta = \dfrac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}$, 4) $\cos\theta = \dfrac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}$, 5) $\tan\theta = \dfrac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}$

6) $1-\cos\theta = 2\sin^2(\theta/2)$, 7) $1+\cos\theta = 2\cos^2(\theta/2)$,

8) $1+\sin\theta = [\cos(\theta/2)+\sin(\theta/2)]^2$, 9) $1-\sin\theta = [\cos(\theta/2)-\sin(\theta/2)]^2$.

Factorisation Formulae:

1) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$, 2) $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$,

3) $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$,

4) $\cos C - \cos D = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right) = -2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$.

Defactorisation Formulae:

1) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$, 2) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$,

3) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$, 4) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$.

Inverse trigonometric functions:

1) $\sin^{-1}(x) = \text{cosec}^{-1}(1/x)$, $\text{cosec}^{-1}(x) = \sin^{-1}(1/x)$,

$\cos^{-1}(x) = \sec^{-1}(1/x)$, $\sec^{-1}(x) = \cos^{-1}(1/x)$,

$\tan^{-1}(x) = \cot^{-1}(1/x)$, $\cot^{-1}(x) = \tan^{-1}(1/x)$.

2) $\sin^{-1}(-x) = -\sin^{-1}(x)$, $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$,

$\tan^{-1}(-x) = -\tan^{-1}(x)$, $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$,

$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$, $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$.

3) $\sin^{-1}(\sin x) = x$, $\cos^{-1}(\cos x) = x$, $\tan^{-1}(\tan x) = x$,

$\cot^{-1}(\cot x) = x$, $\text{cosec}^{-1}(\text{cosec}x) = x$, $\sec^{-1}(\sec x) = x$.

4) $\sin^{-1}x + \cos^{-1}x = \pi/2$, $\tan^{-1}x + \cot^{-1}x = \pi/2$, $\text{cosec}^{-1}x + \sec^{-1}x = \pi/2$.

5) If $x>0, y>0$ and $xy<1$ then, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$.

6) If $x>0, y>0$ and $xy>1$ then, $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$.

7) If $x>0, y>0$ then, $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$.

Trigonometric functions of angles of triangle:

1) In any $\triangle ABC$,

(*i*) $\sin(A+B) = \sin C$, (*ii*) $\sin(B+C) = \sin A$, (*iii*) $\sin(C+A) = \sin B$,

(*iv*) $\cos(A+B) = -\cos C$, (*v*) $\cos(B+C) = -\cos A$, (*vi*) $\cos(C+A) = -\cos B$.

2) In any $\triangle ABC$,

$(i)\sin\left(\frac{A+B}{2}\right)=\cos\left(\frac{C}{2}\right)$, $(ii)\sin\left(\frac{B+C}{2}\right)=\cos\left(\frac{A}{2}\right)$, $(iii)\sin\left(\frac{C+A}{2}\right)=\cos\left(\frac{B}{2}\right)$,

$(iv)\cos\left(\frac{A+B}{2}\right)=\sin\left(\frac{C}{2}\right)$, $(v)\cos\left(\frac{B+C}{2}\right)=\sin\left(\frac{A}{2}\right)$, $(vi)\cos\left(\frac{C+A}{2}\right)=\sin\left(\frac{B}{2}\right)$.

Properties of a triangle:

If A,B,C are angles of ΔABC and a,b,c are the lengths of the sides opposite to the angles A,B,C respectively then,

1) Sine rule: $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

2) Cosine rule: $\cos A=\frac{b^2+c^2-a^2}{2bc}$ *or* $a^2=b^2+c^2-2bc.\cos A$

$\cos B=\frac{c^2+a^2-b^2}{2ca}$ *or* $b^2=c^2+a^2-2ca.\cos B$

$\cos C=\frac{a^2+b^2-c^2}{2ab}$ *or* $c^2=a^2+b^2-2ab.\cos C$

3) Projection rule: $a=b\cos C+c\cos B$, $b=c\cos A+a\cos C$, $c=a\cos B+b\cos A$

4) Tangent rule (Napier's rule):

$\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\left(\frac{C}{2}\right)$, $\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\left(\frac{A}{2}\right)$, $\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\left(\frac{B}{2}\right)$.

Quadratic Equations

1) The equation of the form $ax^2+bx+c=0$, where a,b,c are real numbers and $a\neq 0$ is called quadratic equation.

2) The solution of the quadratic equation is $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

3) $i=j=\sqrt{-1}$ is called an imaginary number. $\therefore\ i^2=j^2=-1$

Progression and Series

1) Sum of the first n natural numbers: $1+2+3+----+n=\sum_{r=1}^{n}r=\frac{n(n+1)}{2}$

2) Sum of the squares of the first n natural numbers: $1^2+2^2+3^2+----+n^2=\sum_{r=1}^{n}r^2=\frac{n(n+1)(2n+1)}{6}$

3) Sum of the cubes of the first n natural numbers: $1^3+2^3+3^3----+n^3=\sum_{r=1}^{n}n^3=\frac{n^2(n+1)^2}{4}=\left[\frac{n(n+1)}{2}\right]^2$

Binomial Theorem

Factorial Notations:

1) n factorial is denoted by n!, 2) $n!=1\times 2\times 3\times -----(n-2)\times(n-1)\times n$, 3) $0!=1$, 4) $1!=1$

Combinations:

1) ${}^nC_r=\frac{n!}{r!(n-r)!}$ where $(r\leq n)$, 2) ${}^nC_r={}^nC_{n-r}$, 3) ${}^nC_0={}^nC_n=1$, 4) ${}^nC_1=n$.

Binomial Theorem:

1) If a,b are real numbers and n is non-negative integer then,

i) $(a+b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + ---------+ {}^nC_r a^{n-r} b^r + -----+ b^n$

ii) $(a-b)^n = a^n - {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 - {}^nC_3 a^{n-3} b^3 + ---------+(-1)^r\, {}^nC_r a^{n-r} b^r + -----+(-1)^n b^n$

2) $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + ------+ x^n$

3) $(1-x)^n = 1 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + ------+(-1)^n x^n.$

Limits

1) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = n\,a^{n-1},$ 2) $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1,$ 3) $\lim_{\theta\to 0} (\cos\theta) = 1,$ 4) $\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1,$

5) $\lim_{x\to 0} (1+x)^{1/x} = e,$ where $e \approx 2.718,$ 6) $\lim_{x\to\infty} (1+1/x)^x = e,$ where $e \approx 2.718,$

7) $\lim_{x\to\infty} \frac{1}{x^n} = 0, \ (n>0)$ 8) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1,$ 9) $\lim_{x\to 0} \frac{\log(1+mx)}{x} = m,$ where m is constant,

10) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log a, \ (a>0, a\neq 1)$ 11) $\lim_{x\to 0} \frac{e^x - 1}{x} = \log_e e = 1.$

Complex Number

Definition - A number of the form z = x + iy where x and y are real numbers and $i = \sqrt{-1}$ is called a complex number.

Note : 1) x is called the real part of z and it is denoted by R(z) or Re(z) and y is called the imaginary part of z and it is denoted by I(z) or Im(z).

2) $i = \sqrt{-1}$ $\therefore i^2 = -1$ 3) i ,j and k are imaginary numbers and $i = j = k = \sqrt{-1}$.

Equality of two complex numbers – Two complex numbers are said to be equal if and only if their real parts are equal and imaginary parts are equal .

Conjugate of complex number- Two complex numbers which differ only in the sign of imaginary part are said to be conjugate of each other.

e.g. a + ib and a – ib are conjugates of each other.

Algebra of complex numbers – Let $z_1 = a+ib$ and $z_2 = c+\ i$ be two complex numbers where $a,b,c,d \in R$ and $i = \sqrt{-1}$. Then addition, subtraction, multiplication and division are performed as follows:

1) Addition: $z_1 + z_2 = (a+ib)+(c+id) \ = (a+c)+i(b+d)$

2) Subtraction: $z_1 - z_2 = (a+ib)-(c+id) \ = (a-c)+i(b-d)$

3) Multiplication: $z_1 z_2 = (a+ib)(c+id) \ = a(c+id)+ib(c+id)$

$= ac + iad + ibc + i^2 bd = (ac-bd)+i(ad+bc) \quad [\because i^2 = -1]$

4) Division: $\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{ac - iad + ibc - i^2 bd}{c^2 - i^2 d^2} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

$[\because i^2 = -1]$

Modulus and argument(amplitude) of a complex number:

1) If z = x + iy is a complex number, then the modulus of z is defined to be $\sqrt{x^2+y^2}$ and is denoted by r or $|z|$ or mod z. $\therefore r=|z|=\sqrt{x^2+y^2}$

2) If z = x + iy is a complex number, then the argument of z is defined as $\tan^{-1}(y/x)$ and is denoted by θ or arg(z) or amp(z) . $\therefore \theta=\tan^{-1}(y/x)$

Powers of a complex number i :

We have $i=\sqrt{-1}, \quad \therefore i^2=-1 \quad \therefore i^3=i^2\times i=-1\times i=-i \quad \therefore i^4=(i^2)^2=(-1)^2=1$

Polar form of a complex number:

If z = x + iy is a complex number, then the polar form of complex number z is given by $z=r(\cos\theta+i\sin\theta)$

Working rule to find amplitude θ:

If z = x + iy is a complex number, then $\alpha=\tan^{-1}\left|\frac{y}{x}\right|$ and we have following cases:

1) If $x>0, y>0$ (I quadrant), then $\theta=\alpha$. 2) If $x<0, y>0$ (II quadrant), then $\theta=\pi-\alpha$.

3) If $x<0, y<0$ (III quadrant), then $\theta=\pi+\alpha$. 4) If $x>0, y<0$ (IV quadrant), then $\theta=2\pi-\alpha$.

Exponential form of a complex number:

If z = x + iy (Cartesian form) is a complex number, then $z=re^{i\theta}$ is called the exponential form of a complex number.

Note: 1) $e^{i\theta}=\cos\theta+i\sin\theta$, 2) $e^{-i\theta}=\cos\theta-i\sin\theta$

Circular Functions: 1) $\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$, 2) $\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$

These formulae are called as Euler's Exponential Functions.

Hyperbolic Functions: 1) $\cosh\theta=\frac{e^{\theta}+e^{-\theta}}{2}$, 2) $\sinh\theta=\frac{e^{\theta}-e^{-\theta}}{2}$

Relation between circular and hyperbolic functions:

1) $\sin i\theta=i\sinh\theta$ 1) $\sinh i\theta=i\sin\theta$

2) $\cos i\theta=\cosh\theta$ 2) $\cosh i\theta=\cos\theta$

3) $\tan i\theta=i\tanh\theta$ 3) $\tanh i\theta=i\tan\theta$

De Moivre's Theorem:

If n is any real number, then one of the values of $(\cos\theta+i\sin\theta)^n$ is $\cos n\theta+i\sin n\theta$.

$$\therefore (\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$$

Roots of a complex number:

If z = x + iy is a complex number, then the polar form of a complex number is,

$$z=x+iy=r(\cos\theta+i\sin\theta)$$

Taking n th root on both sides,

$$z^{1/n}=(x+iy)^{1/n}=r^{1/n}(\cos\theta+i\sin\theta)^{1/n}=r^{1/n}[\cos(2\pi k+\theta)+i\sin(2\pi k+\theta)]^{1/n}$$

$$=r^{1/n}[\cos(2\pi k+\theta)/n+i\sin(2\pi k+\theta)/n] \quad [\because \text{By De Moivre's Theorem}]$$

$$\text{where } k=0,1,2------n-1$$

We put above values to find the roots.

Derivatives

Derivative of standard functions:

1) $\frac{d(x^n)}{dx} = n\,x^{n-1}$, 2) $\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}$, 3) $\frac{d(1/x)}{dx} = -\frac{1}{x^2}$, 4) $\frac{d(x)}{dx} = 1$, 5) $\frac{d(k)}{dx} = 0$ where k is constant.

6) $\frac{d(a^x)}{dx} = a^x \log a, \quad (a > 0)$ 7) $\frac{d(e^x)}{dx} = e^x$, 8) $\frac{d(\log x)}{dx} = \frac{1}{x}$,

9) $\frac{d(\sin x)}{dx} = \cos x$, 10) $\frac{d(\cos x)}{dx} = -\sin x$, 11) $\frac{d(\tan x)}{dx} = \sec^2 x$,

12) $\frac{d(\cot x)}{dx} = -\mathrm{cosec}^2 x$, 13) $\frac{d(\sec x)}{dx} = \sec x \tan x$, 14) $\frac{d(\mathrm{cosecx})}{dx} = -\mathrm{cosecx} \cot x$,

15) $\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$, 16) $\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$, 17) $\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$,

18) $\frac{d(\cot^{-1} x)}{dx} = \frac{-1}{1+x^2}$, 19) $\frac{d(\sec^{-1} x)}{dx} = \frac{1}{x\sqrt{x^2-1}}$, 20) $\frac{d(\mathrm{cosec}^{-1} x)}{dx} = \frac{-1}{x\sqrt{x^2-1}}$.

Integration

Integration of standard functions:

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ where $n \neq -1$, 2) $\int dx = x + c$,

3) $\int \frac{dx}{x} = \log x + c$ where $x > 0$, 4) $\int a^x dx = \frac{a^x}{\log a} + c$ where $a > 0$ and $a \neq 1$, 5) $\int e^x dx = e^x + c$,

6) $\int \sin x dx = -\cos x + c$, 7) $\int \cos x dx = \sin x + c$, 8) $\int \sec^2 x dx = \tan x + c$,

9) $\int \text{cosec}^2 x dx = -\cot x + c$, 10) $\int \text{secx} \tan x dx = \text{secx} + c$, 11) $\int \text{cosecx} \cot x dx = -\text{cosec x} + c$,

12) $\int \tan x dx = \log|\sec x| + c$, 13) $\int \cot x dx = \log|\sin x| + c$,

14) $\int \sec x dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + c$, 15) $\int \text{cosec} x dx = \log|\text{cosec} x - \cot x| + c = \log\left|\tan\left(\frac{x}{2}\right)\right| + c$,

16) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c = -\cos^{-1} x + c$, 17) $\int \frac{dx}{1+x^2} = \tan^{-1} x + c = -\cot^{-1} x + c$,

18) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c = -\text{cosec}^{-1} x + c$,

19) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + c = -\cos^{-1}(x/a) + c$, 20) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a) + c = -\frac{1}{a}\cot^{-1}(x/a) + c$,

21) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}(x/a) + c = -\frac{1}{a}\text{cosec}^{-1}(x/a) + c$,

22) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x + \sqrt{x^2+a^2}\right) + c$, 23) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x + \sqrt{x^2-a^2}\right) + c$,

24) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$, 25) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$,

26) $\int uv dx = u\int v dx - \int \frac{du}{dx}\left[\int v dx\right] dx$

27) $\int \sqrt{a^2-x^2} dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + c$, 28) $\int \sqrt{x^2-a^2} dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x + \sqrt{x^2-a^2}) + c$,

29) $\int \sqrt{x^2+a^2} dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x + \sqrt{x^2+a^2}) + c$,

30) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + c$, 31) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + c$

32) $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$

Standard Substitutions:

Integrand	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$ or $x = a\sin 2\theta$
$\sqrt{\frac{a-x}{x}}$	$x = a\sin^2\theta$ or $x = a\cos^2\theta$
$\sqrt{\frac{a+x}{x}}$	$x = a\tan^2\theta$ or $x = a\cot^2\theta$
$\sqrt{(x-a)(b-x)}$ or $\sqrt{\frac{x-a}{b-x}}$	$x = a\sin^2\theta + b\cos^2\theta$

Definite Integral

Definition of Definite Integral - If f(x) is a function defined on the closed interval [a,b] and $\int f(x)dx = g(x)$, then $\int_a^b f(x)dx =$ g(b) – g(a) where 'a' is called the lower limit and 'b' is called upper limit of integration.

Note - If f(x) and g(x) are two functions and k is a constant, then

1) $\int_a^b [f(x) \pm \text{g(x)}]dx = \int_a^b f(x)dx \pm \int_a^b [\text{g(x)}]dx$

2) $\int_a^b k\, f(x)dx = k\int_a^b f(x)dx$

Properties of Definite Integral

1) If the limits of a definite integral interchanged, then the sign of the integral changes. ,

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

2) Change of independent variable in the given function does not change the value of definite integral.

$$\int_a^b f(x)dx = \int_b^a f(t)dt$$

3) If a < c < b, then a definite integral on an interval [a,b] can be expressed as sum of two definite integrals.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

4) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

5) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$

6) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$

7) $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ if $f(x)$ is even function.

$= 0$ if $f(x)$ is an odd function.

Applications of Definite Integral

1) The area A bounded by the curve y = f(x), X – axis and lying between the lines x = a and x = b is given by,

$$A = \int_{a}^{b} ydx = \int_{a}^{b} f(x)dx.$$

2) The area A bounded by the curve x = g(y), Y – axis and lying between the lines y = c and y = d is given by,

$$A = \int_{c}^{d} xdy = \int_{c}^{d} g(y)dy.$$

3) If the region bounded by the curve y = f(x), the x – axis and the lines x = a and x = b is revolved about the x-axis, then the volume of solid of revolution is given by,

$$V = \pi\int_{a}^{b} y^2 dx = \pi\int_{a}^{b} [f(x)]^2 dx.$$

4) If the region bounded by the curve x = g(y), the y – axis and the lines y = c and y = d is revolved about the y-axis, then the volume of solid of revolution is given by,

$$V = \pi\int_{a}^{b} x^2 dx = \pi\int_{a}^{b} [g(y)]^2 dy.$$

Laplace Transform

1) $L(1)=\frac{1}{s}$ where $s\neq 0$, 2) $L(e^{at})=\frac{1}{s-a}$ where $s\neq a$, 3) $L(\sin at)=\frac{a}{s^2+a^2}$, 4) $L(\cos at)=\frac{s}{s^2+a^2}$,

5) $L(\sinh at)=\frac{a}{s^2-a^2}$ where $s\neq a$, 6) $L(\cosh at)=\frac{s}{s^2-a^2}$ where $s\neq a$, 7) $L(t^n)=\frac{n!}{s^{n+1}}$,

8) $L(e^{at}\sin bt)=\frac{b}{(s-a)^2+b^2}$ where $s>a$, 9) $L(e^{at}\cos bt)=\frac{s-a}{(s-a)^2+b^2}$ where $s>a$,

10) $L(e^{at}\sinh bt)=\frac{b}{(s-a)^2-b^2}$ where $s>a$, 11) $L(e^{at}\cosh bt)=\frac{s-a}{(s-a)^2-b^2}$ where $s>a$,

12) $L(e^{at}t^n)=\frac{n!}{(s-a)^{n+1}}$ where $s>a$,

13) $L^{-1}(\frac{1}{s})=1$ where $s\neq 0$, 14) $L^{-1}\left(\frac{1}{s-a}\right)=e^{at}$ where $s\neq a$,

15) $L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at$, 16) $L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at$,

17) $L^{-1}\left(\frac{1}{s^2-a^2}\right)=\frac{1}{a}\sinh at$, 18) $L^{-1}\left(\frac{s}{s^2-a^2}\right)=\cosh at$,

19) $L^{-1}\left(\frac{1}{s^{n+1}}\right)=\frac{t^n}{n!}$

Fourier Series

Definition of Fourier Series: The Fourier Series for the function f(x) in the interval a < x < a + 2l is given by,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$$

where $a_0=\frac{1}{l}\int_a^{a+2l}f(x)\,dx$ $\quad a_n=\frac{1}{l}\int_a^{a+2l}f(x)\cos(n\pi x/l)\,dx$, $\quad n=1,2,-----,\infty$

$$b_n=\frac{1}{l}\int_a^{a+2l}f(x)\sin(n\pi x/l)\,dx\ ,\quad n=1,2,-----,\infty$$

The formulae for a_0, a_n and b_n are called Euler's formulae.

The real constants $a_0,a_1,a_2---a_n,---,b_1,b_2,----b_n---$ are called Fourier constants of the function f(x).

Note: 1) The Fourier expansion of the function f(x) in the interval (0,2l) is given by,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$$

where $a_0=\frac{1}{l}\int_0^{2l}f(x)\,dx$ $\quad a_n=\frac{1}{l}\int_0^{2l}f(x)\cos(n\pi x/l)\,dx$, $\quad n=1,2,-----,\infty$

$$b_n=\frac{1}{l}\int_0^{2l}f(x)\sin(n\pi x/l)\,dx\ ,\quad n=1,2,-----,\infty$$

2) The Fourier expansion of the function f(x) in the interval $(0,2\pi)$ is given by,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx)$$

where $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$ $\qquad a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos(nx)dx\,,\quad n=1,2,-----,\infty$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(nx)dx \qquad n=1,2,-----,\infty$$

3) The Fourier expansion of the function f(x) in the interval (-l,l) is given by,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$$

where $a_0=\frac{1}{l}\int_{-l}^{l}f(x)\,dx$ $\qquad a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos(n\pi x/l)dx\,,\quad n=1,2,-----,\infty$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin(n\pi x/l)dx\,,\quad n=1,2,-----,\infty$$

4) The Fourier expansion of the function f(x) in the interval $(-\pi,\pi)$ is given by,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx)$$

where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$ $\qquad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx\,,\quad n=1,2,-----,\infty$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)dx\,,\qquad n=1,2,-----,\infty$$

5) $\sin(n\pi)=0$ for all values of n , $\cos(n\pi)=(-1)^n$ when n is odd number and $\cos(n\pi)=1$ when n is even number.

6) If u and v are functions of x, then the Leibnitz rule for successive integration by parts (Generalized rule of integration by parts) is given by,

$$\int uv\,dx=uv_1-u'v_2+u''v_3-u'''v_4+---$$

where dashes denote differentiation and suffixes denote integration.

7) $\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}[a\sin bx-b\cos bx]+c$

8) $\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}[a\cos bx+b\sin bx]+c$

- ***Fourier Series Expansion of Even and Odd Function*:**

  1) Even Function: If f(-x) = f(x), then the function f(x) is called even function.
  2) Odd Function: If f(-x) = - f(x), then the function f(x) is called odd function.

  Note: 1) The Fourier Series of the function f(x) when it is odd function on the interval (-l,l) is given by,

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$$

where $a_0=a_n=0$ and $b_n=\frac{2}{l}\int_0^{l}f(x)\sin(n\pi x/l)dx$.

2) The Fourier Series of the function f(x) when it is an even function on the interval (-l,l) is given by,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)$$

where $a_0=\frac{2}{l}\int_0^{l}f(x)\,dx$, $\quad a_n=\frac{2}{l}\int_0^{l}f(x)\cos(n\pi x/l)dx$ and $b_n=0$.

3) The Fourier Series of the function f(x) when it is odd function on the interval $(-\pi,\pi)$ is given by,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin(nx)$$

Where $a_0 = a_n = 0$ and $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$.

4) The Fourier Series of the function f(x) when it is even function on the interval $(-\pi,\pi)$ is given by,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos(nx)$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$ and $b_n = 0$.

Half Range Fourier Series:

1) Half range Fourier Cosine Series of the function f(x) over (0,l) is given by,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{where} \quad a_0 = \frac{2}{l}\int_0^{l} f(x)dx \;, \quad a_n = \frac{2}{l}\int_0^{l} f(x)\cos(n\pi x/l)dx$$

2) Half range Fourier Cosine Series of the function f(x) over $(0,\pi)$ is given by,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{where} \quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$$

3) Half range Fourier Sine Series of the function f(x) over (0,l) is given by,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{where} \quad b_n = \frac{2}{l}\int_0^{l} f(x)\sin(n\pi x/l)dx.$$

4) Half range Fourier Sine Series of the function f(x) over $(0,\pi)$ is given by,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx.$$

Fourier Transform

<u>Definition of Fourier Transform</u>: The Fourier Transform of the function f(x) is given by,

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

Note: 1) The Fourier Sine Transform of f(x) is given by,

$$F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$$

2) The Fourier Cosine Transform of f(x) is given by,

$$F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx$$