DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATION

An equation containing an independent variable, dependent variable and differentials or differential coefficients is called differential equation.

Following are some examples of differential equations.

1) $xdx + ydy = 0$, 2) $\frac{dy}{dx} + 6y = \sin x$, 3) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = \tan x$

ORDER OF THE DIFFERENTIAL EQUATION [IoPE 2006]

The highest derivative occur in a differential equation is called order of differential equation.

DEGREE OF THE DIFFERENTIAL EQUATION [[IoPE 2006]

The power of the highest order derivative in a differential equation is called degree of a differential equation if the differential equation is free from radicals and fractions.

The differential equation in above example (3) is of second order and first degree.

Examples

State the order and degree of the following differential equations:

1) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right) - 6y = 0$ **[B.T.E. 2008]**

2) $\sqrt{\frac{d^2y}{dx^2}} - \frac{dy}{dx} - xy^2 = 0.$ **[IoPE 2008,2015]**

3) $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 5\frac{d^2y}{dx^2}$ **[BTE 2007]**

Examples for Tutorial

State the order and degree of the following differential equations:

1) $x^2\left(\frac{d^2y}{dx^2}\right)^2 + y\left(\frac{dy}{dx}\right)^3 + x^2 = 0$ [Order = 2, Degree = 2]

2) $\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$ **[B.T.E. 2011]** [Order $= 2$, Degree $= 1$]

3) $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^3}$ [Order =2, Degree = 2]

4) $\frac{d^3y}{dx^3} = \left[K + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$ [Order =3, Degree = 2]

5) $y\left(\frac{dy}{dx}\right)^3 = \frac{x\frac{dy}{dx} + 5}{\frac{dy}{dx}}$ [Order =1, Degree = 4]

FORMATION OF DIFFERENTIAL EQUATIONS

Here we find differential equation from its solution by eliminating arbitrary constants present in it.

We will use following procedure for the formation of differential equation.

1) Find the number of arbitrary constants in the given relation.

2) Differentiate the given relation is equal to the number of times arbitrary constants present in the given relation.

3) The arbitrary constants in the formation of differential equations must be eliminated such as determinant, solving simultaneous equations or by method of substitution etc.

EXAMPLES

A) Form the differential equation by eliminating arbitrary constants from the following relations:

1) $y = ax^2$ **[B.T.E. 2011]** 2) $y = A\sin x + B\cos x$. **[B.T.E. 2013]** 3) $Ae^x + e^y = 1$ **[IoPE 2017]**

B) 1) Form the differential equation for the family of all lines parallel to the line $\frac{x}{2} + \frac{y}{3} = 1$. [IoPE2015]

Examples for Tutorial

A) Form the differential equation by eliminating arbitrary constants from the following relations:

1) $y^2 = 4ax$ **[IoPE 2009]**. 2) $\frac{x}{a} + \frac{y}{b} = 1$ **[IoPE 2012]** $\left[a\frac{dy}{dx} + b = 0 \right]$

3) y = a cos 3t + b sin 3t. **[IoPE2009]** $\left[\frac{d^2y}{dt^2} + 9y = 0 \right]$

B) 1) Form a differential equation of the relation $x = a\sin(\omega t + c)$ where a and c are arbitrary constants. **[B.T.E.92]** $\left[\frac{d^2x}{dt^2} + \omega^2 x = 0 \right]$

2) Find the differential equation of all straight lines having slope unity. $[\frac{dy}{dx} = 1]$

3) Form the differential equations of family of circles with fixed radius r and whose centre lie on x-axis.

[IoPE 2011] $[y^2\left(\frac{dy}{dx}\right)^2 + y^2 = r^2]$

Examples

1) Verify that y = mx is a solution of $(x^2 - 1)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2 + m^2) = 0$. **[IoPE2015]**

2) Verify that y = sin(logx) is a solution of differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$. **[IoPE 2007]**

Examples for Tutorial

1) Verify that $x^2+y^2=r^2$ is a solution of the differential equation $y=x\frac{dy}{dx}+r\sqrt{1+(dy/dx)^2}$ **.[IoPE 2006]**

2) Verify that $y=ax^2+b$ is a solution of $x\left(\frac{dy}{dx}\right)^2-2y\frac{dy}{dx}+ax=0$. **[IoPE 2011]**

3) Verify that y = cosx is a solution of $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$. **[B.T.E. 2010]**

4) Verify that $y=A\sin x+B\cos x$ is a solution of $\frac{d^2y}{dx^2}+y=0$. **[IoPE 2017]**

SOLUTION OF DIFFERENTIAL EQUATION

A solution of a differential equation is the relation between the variables which satisfies the differential equation.

GENERAL SOLUTION OF DIFFERENTIAL EQUATION

The solution which contains as many arbitrary constants as the order of a differential equation is called the general solution of a differential equation.

e.g. $y=A\cos x+B\sin x$ is a general solution where A and B are arbitrary constants.

PARTICULAR SOLUTION OF DIFFERENTIAL EQUATION

The solution which is obtained by giving particular values to the arbitrary constants in the general solution is called particular solution.

e.g. $y=2\cos x+3\sin x$ is a particular solution because the constants 2 and 3 are particular values.

EXAMPLES

1) Find the general solution of:

$(i)\ \frac{dy}{dx}=3e^{3x}+2,\quad (ii)\ \frac{d^2y}{dx^2}=12x^2+6x$ **[IoPE 2014]**

2) Find the particular solution of following differential equations:

$(i)\ \frac{dy}{dx}=6-3x,$ and $y=5$ given at $x=0$ and $y=0$ **[B.T.E.2008]**

$(ii)\ \frac{dy}{dx}=3x^2-2x+5$ and $y=5$ when $x=2$ **[B.T.E.1985]**

Examples for Tutorial

1) Find the general solution of $\frac{d^3y}{dx^3}=e^x$. $\left[y=e^x+c_1x^2+c_2x+c_3\right]$

2) Solve the differential equation $\frac{dx}{dt}=t+4,$ given that $x=0$ at $t=0$. Also find x when t = 3. **[B.T.E. 1985]**

$\left[x=\frac{t^2}{2}+4t,\ x=16.5\right]$

3) Solve the differential equation $\frac{dx}{dt}=6-3t,$ if $x=0$ when $t=0$. $\left[x=6t-\frac{3t^2}{2}+c,\ x=6t-\frac{3t^2}{2}\right]$

Methods of solving differential equations of first order and first degree

1) Variable Separable Method: The differential equation of the form f(x) dx - f(y) dy = 0 can be written as f(x) dx = f(y) dy where x and y are separated.

Integrating both sides we get,

$$\int f(x)dx = \int f(y)dy + c$$

where 'c' is an arbitrary constant.

Note - A differential equation of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to variable separable form by the substitution ax + by + c = 0.

Examples

Solve the following differential equations:

1) $\frac{y}{x}\frac{dy}{dx} = \sqrt{1 + x^2 + y^2 + x^2y^2}$.**[IoPE 2013]**

2) $\sec^2 x \tan y .dx + \sec^2 y \tan x .dy = 0$ when $x = y = \pi/4$ **[IoPE2006]**

3) $1 + \frac{dy}{dx} = \text{cosec}(x + y)$ **[IoPE 2015]**

Examples for Tutorial

Solve the following differential equations:

1) $(x + y)^2 \frac{dy}{dx} = a^2$ $\left[y - a\tan\left(\frac{x + y}{a}\right) = c \right]$

2) $(x - y)^2 \frac{dy}{dx} = a^2$ **[B.T.E.'91]** $\left[y + c = \frac{a}{2}\log\left(\frac{x - y - a}{x - y + a}\right) \right]$

3) $\sin x \cos y\, dy + \sin y \cos x\, dx = 0$. **[IoPE 2009]** $[\sin x \sin y = c]$

4) $\tan y \frac{dy}{dx} = \sin(x + y) - \sin(x - y)$ **[IoPE 2017]** $[\log(\sec y + \tan y) = 2\sin x + c]$

5) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$. **[IoPE 2006]** $\left[e^y = e^x + \frac{x^3}{3} + c \right]$

HOMOGENEOUS DIFFERENTIAL EQUATIONS

A differential equation is said to be homogeneous differential equation if the degree of every term in the differential equation is same.

Note: (a) All homogeneous differential equations are solved by using following substitutions.

Put $y = vx \qquad \therefore v = y/x$

Differentiating with respect to x on both sides,

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

(b) If a differential equation contains the term of type (y/x), then it is also solved by using homogeneous differential equation method.

Examples

Solve the following differential equations:

1) $(x^2 - y^2)dx = 2xydy$ **[B.T.E. 2007,IoPE 2016]**

2) $xy\frac{dy}{dx} = (x+y)^2$ **[B.T.E.91, IoPE 2016]**

3) $[x\tan(y/x) - y\sec^2(y/x)]dx - x\sec^2(y/x)dy = 0$ **[IoPE 2012]**

Examples for Tutorial

Solve the following differential equations:

1) $x + y\frac{dy}{dx} = 2y$ **[IoPE2011]** $\qquad \left[\log(y-x) = c + \frac{x}{y-x}\right]$

2) $\frac{dy}{dx} = \frac{4x-3y}{3x-2y}$ **[IoPE2012]** $\qquad [y^2 - 3xy + 2x^2 = c]$

3) $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$ $\qquad [x(c+y) = ay^2]$

4) $x(x+y)\,dy - y^2dx = 0$ **[IoPE 2016]** $\qquad \left[\frac{y^2}{2x^2} + \log x = c\right]$

5) $\left(x\frac{dy}{dx} - y\right)e^{y/x} = x^2\cos x$ **[IoPE2017]** $\qquad [e^{y/x} = \sin x + c]$

LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together.

Note: (a) The general form of linear differential equation is $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x or constants.

(b) The solution of the general form of linear differential equation is given by,

$$ye^{\int Pdx} = \int Q e^{\int Pdx}dx + c.$$

(c) $e^{\int Pdx}$ is called an integrating factor. $\quad \therefore$ Integrating factor $= e^{\int Pdx}$ $\qquad$ (d) $e^{\log x} = x$

Examples

Solve the following differential equations:

1) $\frac{dy}{dx} + y\cot x = \cos x$ **[IoPE 2016,B.T.E.'85]**

2) $\cos^2 x\frac{dy}{dx} + y = \tan x$ **[IoPE 2006]**

3) $x\log x\frac{dy}{dx} + y = (\log x)^2$ **[IoPE2013]**

Examples for Tutorial

Solve the following differential equations:

1) $\frac{dy}{dx} + y\tan x = \cos^2 x$ **[IoPE2017]** $\qquad [y\sec x = \sin x + c]$

2) $\frac{dy}{dx} + \frac{4x}{x^2+1}y = \frac{1}{(x^2+1)^3}$ $\qquad \left[y(x^2+1) = \tan^{-1}x + c\right]$

3) $x^2\frac{dy}{dx} = 3x^2 - 2xy + 1$ **[IoPE2014]** $\qquad \left[x^2 y = x^3 + x + c\right]$

4) $(1-x^2)\frac{dy}{dx} + x\,y = x$ **[IoPE2014]** $\qquad \left[y = 1 + c\sqrt{1-x^2}\right]$

5) $(x^2+1)\frac{dy}{dx} + 2xy = \frac{1}{x^2+1}$ **[IoPE2016]** $\qquad [y\,(x^2+1) = \tan^{-1}x + c]$

Second order Differential Equations

There are two cases to study differential equations of second order.

Case I) $\frac{d^2y}{dx^2} = f(x)$

In this type, we integrate above differential equation two times to get the required solution. The initial conditions are used to find the values of arbitrary constants.

Case II) $\frac{d^2y}{dx^2} = f(y)$.

In this case, multiply above differential equation by $2\frac{dy}{dx}$, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2f(y)\frac{dy}{dx} ---(1) \qquad \text{But } \frac{dy^2}{dx} = 2y\frac{dy}{dx}, \qquad \text{Put } y = \frac{dy}{dx}$$

$$\therefore \frac{d(dy/dx)^2}{dx} = 2\frac{dy}{dx}\frac{d(dy/dx)}{dx} = 2\frac{dy}{dx}\frac{d^2y}{dx^2} \qquad \therefore (1) \text{ becomes,}$$

$$\therefore \frac{d(dy/dx)^2}{dx} = 2f(y)\frac{dy}{dx}$$

Integrating with respect to x on both sides,

$$\therefore \int \frac{d(dy/dx)^2}{dx}dx = 2\int f(y)\frac{dy}{dx}dx + c \qquad \therefore (dy/dx)^2 = 2\int f(y)dy + c$$

This is integrated by usual method.

Examples

1) Find the equation of the curve passing through the points (0,4) and (2,0) for which $\frac{d^2y}{dx^2} = 4x.$.

[IoPE2016]

2) Solve the differential equation $\frac{d^2y}{dx^2} = 4y$ given that y = 2 and $\frac{dy}{dx} = 2$ when x = 0. **[IoPE2017]**

Examples for Tutorial

Solve the following differential equations:

1) $\frac{d^2y}{dx^2} = 3x^2$, given that $y = \frac{3}{4}$ when x = 0 and y = 2 when x = 1. **[IoPE2009, B.T.E.'88]**

$\left[4y = x^4 + 4x + 3\right]$

2) Solve the differential equation $\frac{d^2y}{dx^2} = 2y^3$ given $\frac{dy}{dx} = 0$ when y = 0 and y = 1 when x = 1. **[IoPE2012]**

APPLICATION OF DIFFERENTIAL EQUATIONS TO SIMPLE ELECTRICAL CIRCUITS

A circuit is made up of three passive components – resistance (R), capacitance (C) and inductance (L). There is an active element voltage source which is battery or a generator (E).

BASIC RELATIONS

1) Current – Current is the rate of flow of electricity.

If i is the current and q is the charge then, $i = \frac{dq}{dt}$ or $q = \int i\, dt$.

2) Voltage drop across resistance R = Ri (Ohm's Law)

3)Voltage drop across inductance L = $L = \frac{di}{dt}$.

4) Voltage drop across capacitance C = $\frac{q}{C}$.

KIRCHHOFF'S LAWS

1) The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.

2) The algebraic sum of the currents flowing into (or from) any node is zero.

DIFFERENTIAL EQUATION OF R-L SERIES CIRCUIT: $Ri + L\frac{di}{dt} = E \ or \ \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$.

DIFFERENTIAL EQUATION OF R-L-C SERIES CIRCUIT t: $Ri + L\frac{di}{dt} + \frac{q}{C} = E \ or \ \frac{di}{dt} + \frac{R}{L}i + \frac{q}{LC} = \frac{E}{L}$.

Examples

1) A resistance of 100 ohm, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at t = 0.5 sec. if i = 0 at t = 0. **[IoPE 2016]**

2) A 20 ohm resistance is connected in series with a capacitor of of 0.01 farads and the electronic force E volts is given by $40e^{-3t} + 20e^{-6t}$. If q = 0 at t = 0. Show that the maximum charge on capacitor is 0.25 coulombs. **[IoPE 2016]**

Examples for Tutorial

1) A constant e.m.f. E is introduced into an L-R circuit The differential equation of the circuit is $E - L\frac{di}{dt} = Ri$.
Find the current at any time 't' given that i = 0 when t = 0 **[IoPE2009]** $\left[i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)\right]$

2) The discharge of a condenser of capacity C through a large resistance R obeys the law $\frac{dq}{dt} = -\frac{q}{CR}$,
q being the amount of charge at any time t. Find the charge q after t seconds if the initial charge is q_0.
[IoPE.2012] $\left[q = q_0 . e^{-t/RC}\right]$

3) A flexible rope of negligible weight is wound just once round a circular post. The tension T and the angle of wrap θ are given by $\frac{dT}{d\theta} = \mu T$, where μ is coefficient of friction . Find T if θ = 2π and μ = 0.5.
Given that T = 10 and if θ = 0. **[IoPE2012]**