Derivative

Definition of Derivative of a function

If $y = f(x)$ is a function of x and h is an increment in x then,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) --- (1)$$

is called derivative of y with respect to x, provided the limit exists.

Note: The formula in (1) is also called as the derivative of the function by using the **First Principle.** It is also called as **Leibnitz's rule**.

Examples

Find the derivative of following functions by using the First Principle.

1) $x^n \quad x \in R.$ 2) sinx 3) a^x

Examples for Tutorial

Find the derivative of following functions by using the First Principle.

1) cosx **[B.T.E.2016]** 2) tanx **[IoPE2012]** 3) cosecx

4) logx **[IoPE .2010]** 5) e^x **[IoPE.2010]**

Derivative of standard functions (Formulae)

1) $\frac{d(x^n)}{dx} = n\,x^{n-1},$ 2) $\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}},$ 3) $\frac{d(1/x)}{dx} = -\frac{1}{x^2},$ 4) $\frac{d(x)}{dx} = 1,$ 5) $\frac{d(k)}{dx} = 0$ where k is constant.

6) $\frac{d(a^x)}{dx} = a^x \log a, \quad (a > 0)$ 7) $\frac{d(e^x)}{dx} = e^x,$ 8) $\frac{d(\log x)}{dx} = \frac{1}{x},$

9) $\frac{d(\sin x)}{dx} = \cos x,$ 10) $\frac{d(\cos x)}{dx} = -\sin x,$ 11) $\frac{d(\tan x)}{dx} = \sec^2 x,$

12) $\frac{d(\cot x)}{dx} = -\text{cosec}^2 x,$ 13) $\frac{d(\sec x)}{dx} = \sec x \tan x,$ 14) $\frac{d(\text{cosec}x)}{dx} = -\text{cosec}x \cot x,$

15) $\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}},$ 16) $\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}},$ 17) $\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2},$

18) $\frac{d(\cot^{-1} x)}{dx} = \frac{-1}{1+x^2},$ 19) $\frac{d(\sec^{-1} x)}{dx} = \frac{1}{x\sqrt{x^2-1}},$ 20) $\frac{d(\text{cosec}^{-1} x)}{dx} = \frac{-1}{x\sqrt{x^2-1}}.$

$$\text{Note: If } \frac{d[f(x)]}{dx} = g(x), \text{ then } \frac{d[f(ax)]}{dx} = a.g(ax).$$

Rules of differentiation:

1) **Derivative of a sum or difference**: If u and v are differentiable functions of x and $y = u \pm v$, then

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}.$$

2) **Derivative of a product**: If u and v are differentiable functions of x and $y = uv$, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

3) **Derivative of a quotient**: If u and v are differentiable functions of x and $y = u/v$ where $v \neq 0$ then,

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Examples

Find the derivative of following functions or find $\frac{dy}{dx}$.

1) $x\tan^{-1}x$ **[IoPE2015]**

2) $y = e^x(\sin x + \cos x)$ **[IoPE 2016]**

3) $\frac{x}{(x-1)(x+2)}$ **[IoPE 2008]**

4) $\frac{x\sin x}{\log x + e^x}$ **[IoPE 2013]**

Examples for Tutorial

Find the derivative of following functions or find $\frac{dy}{dx}$.

1) $e^x \sin x - \tan x$ **[IoPE2008]** $\left[\frac{dy}{dx} = e^x(\cos x + \sin x) - \sec^2 x\right]$

2) $x^2 \sin x + 3^x \tan x - \log_{10} x$ **[IoPE 2010]**

$\left[\frac{dy}{dx} = x^2 \cos x + 2x\sin x + 3^x \sec^2 x + tanx\ 3^x \log 3 + \frac{1}{x\log 10}\right]$

3) $\frac{x^3+1}{2x^2-1}$ **[IoPE 2009, B.T.E.'97]** $\left[\frac{dy}{dx} = \frac{2x^4 - 3x^2 - 4x}{(2x^2-1)^2}\right]$

4) $\frac{1+\log x}{1-\log x}$ $\left[\frac{dy}{dx} = \frac{2}{x(1-\log x)^2}\right]$

Derivative of a composite function

If y is a function of u and u is a function of x, then y is a function of x. So, function of a function is called a composite function.

Chain Rule:

If y is a differentiable function of u and u is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

EXAMPLES: Find the derivative of following functions by using chain rule.

1) $(3-2x)^7$ **[IoPE2010]** 2) $\sin x^0$ **[IoPE2015]**

Examples for Tutorial

Find the derivative of following functions by using chain rule.

1) $\sqrt[3]{2x+3}$ $\left[\frac{dy}{dx}=\frac{1}{3}(2x+3)^{-2/3}\right]$ 2) $(4x-7)^6$ **[IoPE 2015]** $\left[\frac{dy}{dx}=24(4x-7)^5\right]$

3) $\tan(x^0+45^0)$ **[B.T.E .'90]** $\left[\frac{dy}{dx}=\frac{\pi}{180}\sec^2(x^0+45^0)\right]$

Derivative of trigonometric functions

1) $\frac{1-\cos 2x}{\sin 2x}$ 2) $\sin^{-1}(2x\sqrt{1-x^2})$ **[IoPE 2016]** 3) $\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$ **[B.T.E.'85]**

Examples for Tutorial

Find $\frac{dy}{dx}$

1) $\frac{\sin 2x}{1+\cos 2x}$ $\left[\frac{dy}{dx}=\sec^2 x\right]$ 2) $\sqrt{\frac{1+\cos 2x}{1-\cos 2x}}$ $\left[\frac{dy}{dx}=-\text{cosec}^2 x\right]$

3) $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ **[IoPE 2013]** $\left[\frac{dy}{dx}=\frac{-2}{1+x^2}\right]$ 4) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ $\left[\frac{dy}{dx}=\frac{2x}{1+x^2}\right]$

Derivative of inverse trigonometric functions

If y = f(x) is a differentiable function of x such that the inverse function $x=f^{-1}(y)$ exists, then x is a differentiable function of y and

$$\frac{dx}{dy}=\frac{1}{dy/dx}, \text{ where } \frac{dy}{dx}\neq 0.$$

Examples

Find $\frac{dy}{dx}$ for the following.

1) $\sin^{-1}x$ **[IoPE 2017]** 2) $\tan^{-1}x$ **[IoPE 2009]**

3) $\cos^{-1}\left(\frac{\cos x+\sin x}{\sqrt{2}}\right)$ **[IoPE 2013]** 4) $\tan^{-1}\left(\frac{\sqrt{1+\cos 2x}}{\sqrt{1-\cos 2x}}\right)$ **[IoPE 2006]**

Examples for Tutorial

Find $\frac{dy}{dx}$ for the following.

1) $\cos^{-1}x$ $\left[\frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}}\right]$

2) $\cot^{-1}x$ **[IoPE 2009]** $\left[\frac{dy}{dx}=\frac{-1}{1+x^2}\right]$

3) $\operatorname{cosec}^{-1}x$ $\left[\frac{dy}{dx}=\frac{-1}{x\sqrt{x^2-1}}\right]$

4) $\sin^{-1}\left(\frac{\cos x+\sin x}{\sqrt{2}}\right)$ $\left[\frac{dy}{dx}=-1\right]$

5) $\cos^{-1}\left(\frac{\cos x+\sin x}{\sqrt{2}}\right)$ **[IoPE2013]** $\left[\frac{dy}{dx}=-1\right]$

Logarithmic Differentiation

The differentiation of the function of the form $y=[f(x)]^{g(x)}$ i.e function raised to function is called logarithmic differentiation.

Note: This method is also useful to find the derivative of a function expressed as the product or division of number of functions.

Examples

Differentiate with respect to x.

1) $\log\left[e^{3x}.\sqrt{\frac{x+1}{x-5}}\right]$ **[IoPE 2014]**

2) x^x **[IoPE 2011]**

3) $(\sin x)^{\tan x}$ **[IoPE 2010]**

4) $x^x+(\sin x)^x$. **[IoPE 2013]**

Examples for Tutorial

Find dy/dx for the following.

A)

1) $\log\left[e^x\left(\frac{2x-3}{3x+5}\right)^{3/4}\right]$ **[IoPE 2006]** $\left[\frac{dy}{dx}=1+\frac{3}{4}\left[\frac{2}{2x-3}-\frac{3}{3x+5}\right]\right]$

2) $(\sin x)^{\log x}$ **[IoPE 2008]** $\left[\frac{dy}{dx}=(\sin x)^{\log x}\left[\cot x\cdot\log x+\frac{\log(\sin x)}{x}\right]\right]$

8) $(\log x)^{\log x}+x^{\sin x}$ **[IoPE 2011]** $\left[\frac{dy}{dx}=\frac{(\log x)^{\log x}}{x}[1+\log(\log x)]+x^{\sin x}[\frac{\sin x}{x}+\log x(\cos x)]\right]$

B) 1) If $x^p y^q=(x+y)^{p+q}$, then show that $\frac{dy}{dx}=\frac{y}{x}$.**[IoPE 2012, BTE 2017]**

Derivative of implicit functions

A relation in which x and y are not separated easily is called an implicit function. The derivative of such function is called implicit differentiation.
The function of the form f(x,y) = 0 is called an implicit function.
e.g. $x^5 y^4 = (x+y)^9$ is an implicit function.

Procedure to solve examples

1) Differentiating x with respect to x is 1.
2) Derivative of y with respect to x is $\frac{dy}{dx}$. We should always remember to write $\frac{dy}{dx}$ while differentiating a function of y.
3) Collect the coefficients of $\frac{dy}{dx}$ on L.H.S.
4) Then find the value of $\frac{dy}{dx}$.

Examples:

Differentiate with respect to x or find $\frac{dy}{dx}$.

1) $x^{2/3} + y^{2/3} = a^{2/3}$ **[IoPE .2016, B.T.E.'92]**

2) **x**y = log(xy). **[IoPE 2015]**

3) $x.\log y + y.\log x = 0$ **[IoPE .2008]**

Examples for Tutorial

A) Find $\frac{dy}{dx}$ for the following:

2) $x^2 + 2xy + y^2 = 42$ **[B.T.E.'79]** $\left[\frac{dy}{dx} = -\frac{2(x+y)}{2x+3y^2}\right]$

3) $x^3 + y^3 = 3\,axy$ **[IoPE 2010]** $[\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}]$

4) $\sin y = \log(x+y)$, **[IoPE 2007]** $\left[\frac{dy}{dx} = \frac{1}{[(x+y)\cos y - 1]}\right]$

B)

4) If siny = x sin(a + y), prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$. **[IoPE 2007, B.T.E.'86]**

5) If $x^y = e^{x-y}$ **,** show that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$. **[B.T.E.'85,87,IoPE 2010]**

Derivative of parametric functions

If x = f(t) and y = g(t) are two differentiable functions of t such that y is defined as a function of x, then

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt},\ \text{where}\ \frac{dx}{dt}\neq 0.$$

Examples

Find $\frac{dy}{dx}$. for the following:

1) $x = at^2$ and $y = 2at$, **IIoPE 2012]** 2) $x=\frac{e^{2t}+1}{2}$ and $y=\frac{e^{2t}-1}{2}$, [**IoPE2016]**

3) If x $= a(\theta+\sin\theta), y = a(1-\cos\theta)$, find $\frac{dy}{dx}$ at $\theta=\pi/2$.**[IoPE2010]**

Examples for Tutorial

A) Find $\frac{dy}{dx}$ if

1) $x=a\cos^3\theta$ and $y=b\sin^3\theta$ **[IoPE2015]** $\left[\frac{dy}{dx}=-\frac{b}{a}\tan\theta\right]$

2) **5)** $x=\sqrt{\cos 2\theta}$ and $y=\sqrt{\sin 2\theta}$ **[IoPE 2008]** **[** $\frac{dy}{dx}=-\frac{x^3}{y^3}$]

B)

1) If $x = a(1-\cos^3\theta)$and$y = a\sin^3\theta$, then show that $\frac{dy}{dx}=\tan\theta$ **.[IoPE 2013]**

2) If $x=3\cos\theta$ and $y=\sqrt{3}\sin\theta$, prove that $\frac{dy}{dx}=-1$ at $\theta=\pi/6$. **[IoPE 2014]**

Derivative of one function with respect to other

Let $u=f(x)$ and $v=g(x)$ be two functions, then the derivative of u with respect to v is given by,

$$\frac{du}{dv}=\frac{du/dx}{dv/dx}$$

Examples

1) Differentiate $\tan 2x$ with respect to $\log 3x$.

2) Differentiate $\tan^{-1}9x$ with respect to e^{6x}.

Examples for Tutorial

1) Differentiate $\sin^{-1}x$ with respect to $\sqrt{1-x^2}$. $[-1/x]$

2) Differentiate $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$ with respect to $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$. $[-2]$

Second Order Derivatives:

1) $\frac{dy}{dx}$ is called first order derivative.

2) $\frac{d(dy/dx)}{dx} = \frac{d^2y}{dx^2}$ is called second order derivative.

So, derivative of derivative is called second order derivative.

Examples:

1) If $y = 2\sin 2x - 5\cos 2x$, show that $\frac{d^2y}{dx^2} + 4y = 0$.**[IoPE 2008]**

2) If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - y/x = 0$. **[IoPE 2013]**

Examples for Tutorial

1) If $y = 2\sin 2x - 5\cos 2x$, show that $\frac{d^2y}{dx^2} + 4y = 0$.**[IoPE 2008]**

2) If $y = \left(x + \sqrt{x^2+1}\right)^m$, show that $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$.

3) If If $y = e^{\tan^{-1}x}$ show that $(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$

4) If If $y = e^{m\sin^{-1}x}$ show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

Applications of Derivative

Tangents and Normals

1) Equation of tangent to the curve y = f(x) at the point (x_1,y_1) is given by,

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1).$$

2) Equation of normal to the curve y = f(x) at the point (x_1,y_1) is given by,

$$y - y_1 = \frac{-1}{(dy/dx)_{(x_1,y_1)}}(x - x_1).$$

Example

1) Find the equation of tangent and normal to the curve $y=5x^2-2x+3$ at (1,6). **[IoPE 2016]**

Examples for Tutorial

1) Find the equation of tangent and normal to the curve $y=5x^2-2x+3$ at (1,6).**[IoPE 2016]**

[Tangent : 8x – y – 2 =0, Normal : x + 8y – 49 = 0]

2) Find the points on the curve $y=12x-x^3$ at which the slope is zero. **[(2,16), (-2,-16)]**

3) Find the slope of tangent to the curve $x^2+y^2=25$ at the point (-3,4). **[B.T.E. 2017] [Slope = 3/4]**

Maxima and Minima:

1) A function $y=f(x)$ has a maximum at $x=a$ if $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}<0$ and its maximum value is $f(a)$.

2) A function $y=f(x)$ has a minimum at $x=a$ if $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}>0$ and its minimum value is $f(a)$.

Turning point or stationary point

The point at which the function changes its nature is called turning point or stationary point.

Point of inflexion or saddle point

The point at which $\frac{dy}{dx}=0$ is called point of inflexion or saddle point.

Examples:

Find the maximum and mnimum values of the following:

1) $x(k-x)$. [**IoPE 2011]** 2) $4-x-x^2$. **[IoPE 2013]**

3) Divide 70 into two parts so that the sum of their squares is minimum.[**IoPE 2015]**

Examples for Tutorial

A) Find maximum/minimum values of the following functions.

1) x^3-9x^2+24x **[IoPE2008**, **B.T.E. 2015,17]**

[y = 16 is minimum value at x = 4 and y = 20 is maximum value at x = 2]

4) $\frac{\log x}{x}$ $\left[e \text{ max.}, Y\text{ max.} = \frac{1}{e}\right]$

B) 1) Divide 80 into two parts such that their product is maximum. [**B.T.E2015,16] [40 , 40]**

2) A rod 72 cm long is bent to form a rectangle. Find the dimensions when its area is to be maximum.
[IoPE 2013] **[$x = 18$ cm. $y = 18$ cm.]**

Radius of curvature

The reciprocal of curvature of a curve at any point is called the radius of curvature at that point and it is denoted by ρ (rho).

Note: If $y = f(x)$ is a function, then the radius of curvature is denoted by ρ (rho) and is given by,

$$\rho = \frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2}$$

Examples

1) Find the radius of curvature of the curve $y = \log(\sin x)$ at $x = \pi/2$.

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2) Find the radius of curvature of the parabola $x = at^2,\ y = 2at$ at any point t.

Examples for Tutorial

1) Find the radius of curvature of e^x at the point (0,1). $[\rho = 2\sqrt{2}]$

2) Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point (1,1). $[\rho = \sqrt{2}/3]$