Definite Integral

Definition of Definite Integral - If f(x) is a function defined on the closed interval [a,b] and $\int f(x)dx = g(x)$, then $\int_a^b f(x)dx =$ g(b) – g(a) where 'a' is called the lower limit and 'b' is called upper limit of integration.

Rules of definite integral

If f(x) and g(x) are two functions , then

1) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b [g(x)]dx$

2) If k is constant then, $\int_a^b k\, f(x)dx = k\int_a^b f(x)dx$ where k is constant.

Examples

1) $\int_0^1 \frac{3x^3 - 4x^2 + 1}{\sqrt{x}} dx$ **[B.T.E.'90]**

2) $\int_0^4 \frac{x^2 - 4}{x^2 + 4} dx$

3) $\int_0^{\pi/4} \left(\sec^2 x + \cos 2x\right) dx$

4) Find the value of k if $\int_0^1 (3x^2 + 2x + k)\, dx = 0$. **[B.T.E.'88]**

Examples for Tutorial

Evaluate

1) $\int_0^2 \frac{dx}{\sqrt{x+2} - \sqrt{x}}$ **[B.T.E.'88]** $[I = 8/3]$

2) $\int_0^1 \frac{1-x^2}{1+x^2} dx$ $\left[I = \frac{\pi}{2} - 1\right]$

3) $\int_0^\infty \frac{dx}{a^2 + b^2x^2}$ **[B.T.E.'92]** $\left[I = \frac{\pi}{2ab}\right]$

4) $\int_0^{\pi/4} \sin 2x . \sin 3x\, dx$ $\left[I = \frac{1}{2}\left(1 + \frac{1}{5\sqrt{2}}\right)\right]$

5) $\int_1^e \log x\, dx$ **[B.T.E.'91]** [I = 1]

6) Find the value of a if $\int_1^a (3x^2 + 2x + 1)\, dx = 11$. $[a = 2]$

Definite integral by the method of substitution

Some integrals cannot be solved by standard formulae. We can solve such integrals by the method of substitution. We replace the new variable by old variable while substitution. When we use substitution in integrals, we also change the limits of integration.

Examples

1) $\int_0^4 \frac{dx}{x+\sqrt{x}}$ **[B.T.E.'92]** 2) $\int_0^2 \frac{5x\ dx}{x^2+4}$ **[B.T.E.'86]** 3) $\int_0^{\pi/2} \sin x \cos^2 x\, dx$

Examples for Tutorial

1) $\int_0^1 \frac{x^2}{1+x^6} dx$ $\left[I = \frac{\pi}{16} \right]$

2) $\int_0^{\pi} \sin x \cos^3 x\, dx$ $[I = 0]$

3) $\int_0^1 \frac{(\tan^{-1} x)^2}{1+x^2} dx$ $\left[I = \frac{\pi^3}{192} \right]$

4) $\int_1^e \frac{\log x}{x} dx$ **[B.T.E.'91]** [I = 1/2]

Properties of Definite Integral

1) If the limits of a definite integral interchanged, then the sign of the integral changes. ,

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

2) Change of independent variable in the given function does not change the value of definite integral.

$$\int_a^b f(x)dx = \int_b^a f(t)dt$$

3) If a < c < b, then a definite integral on an interval [a,b] can be expressed as sum of two definite integrals.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

4) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

5) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$

6) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$

7) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even function.

$= 0$ if $f(x)$ is an odd function.

Note – When we use the substitution in definite integral, then we also change the limits of integration.

Examples on the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

1) $\int_0^3 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}dx$

2) $\int_0^{\pi/2} \frac{\sqrt[3]{\sec x}}{\sqrt[3]{\sec x}+\sqrt[3]{\operatorname{cosec} x}}dx$ **[IoPE 2008]**

3) $\int_0^{\pi/2} \log\left(\frac{1+\sin x}{1+\cos x}\right)dx$. **[IoPE 2016]**

Examples for Tutorial

1) $\int_0^9 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}dx$ $\left[I=\frac{9}{2}\right]$

2) $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$ $\left[I=\frac{\pi}{8}\log 2\right]$

3) $\int_0^{\pi/2} \frac{dx}{1+\sqrt[n]{\cot x}}$ **[B.T.E.'2008]** $\left[I=\frac{\pi}{4}\right]$

4) $\int_0^{\pi/4} \log(1+\tan x)\,dx$. [**IoPE 2013,B.T.E.'89]** $\left[I=\frac{\pi}{4}\right]$

Examples on the property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$

1) $\int_3^7 \frac{(10-x)^2}{x^2+(10-x)^2}dx$

2) $\int_{\pi/6}^{\pi/3} \cos^2 x\,dx$ **[IoPE 2015]**

3) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\cot x}$ **[IoPE2008]**

Examples for Tutorial

1) $\int_2^5 \frac{\sqrt{x}}{\sqrt{7-x}+\sqrt{x}}\,dx$ **[IoPE 2011]** $\left[I=\frac{3}{2}\right]$

2) $\int_{\pi/6}^{\pi/3} \sin^2 x\,dx$ $\left[I=\frac{\pi}{12}\right]$

3) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$ **[B.T.E.'85]** $\left[I=\frac{\pi}{12}\right]$

4) $\int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}dx$ $\left[I=\frac{\pi}{12}\right]$

5) $\int_{\pi/6}^{\pi/3} \frac{\sec x}{\sec x+\operatorname{cosec} x}dx$ **[IoPE 2016]** $\left[I=\frac{\pi}{12}\right]$

Examples on even and odd functions

1) $\int_{-1}^{1} \frac{x+x^2}{1+x^2}\,dx$ **[IoPE 2006]** 2) $\int_{-\pi/4}^{\pi/4} \sin^2 x \sec^4 x\,dx$ 3) $\int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin x}{2+\sin x}\right)dx$

Examples for Tutorial

1) $\int_{-1}^{1} \frac{x^2}{1+x^2}\,dx$ $\left[I = 2-\frac{\pi}{2}\right]$ 2) $\int_{-\pi/4}^{\pi/4} \sin^2 x\,dx$ $\left[I = \frac{\pi-2}{8}\right]$

3) $\int_{-1}^{1} \frac{x^{11}}{\sqrt{1-x^2}}\,dx$ [I = 0] 4) $\int_{-\pi/4}^{\pi/4} \text{cosec}^2 x\,dx$ [I = 0] 5) $\int_{-\pi/2}^{\pi/2} \frac{\sin^3 x}{1+\cos^2 x}\,dx$ [I = 0]

Applications of Definite Integral

Area as definite integral

1) The area A bounded by the curve y = f(x), X – axis and lying between the lines x = a and x = b is given by,

$$A = \int_a^b y dx = \int_a^b f(x) dx.$$

2) The area A bounded by the curve x = g(y), Y – axis and lying between the lines y = c and y = d is given by,

$$A = \int_c^d x dy = \int_c^d g(y) dy.$$

Examples

1) Find the area bounded by the curve $y = 2x - x^2$ and the x – axis. **[IoPE.2014]**

2) Find the area under the curve y = sinx from x = 0 to x = 2π. **[IoPE 2007, B.T.E'86]**

3) Find by integration the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **[IoPE 2008,B.T.E.'91]**

Examples for Tutorial

1) Find the area bounded by the curve $y = 4 - x^2$ and the x-axis. **[IoPE2007,B.T.E.'2011]** $\left[A = \frac{32}{3} \text{square units}\right]$

2) Find the area of the region bounded by $x^2 = 16y$, y = 1 , y = 4 and the y – axis in the first quadrant. $\left[A = \frac{56}{3} \text{square units}\right]$

3) Find the area of the circle $x^2 + y^2 = 9$. **[IoPE 2016]** $[A = 9\pi$ square units]

Area between two curves

If $y_1 = f(x)$ and $y_2 = g(x)$ are two functions from $x = a$ to $x = b$, then area between two curves is given by,

$$A = \int_a^b \left[f(x) - g(x) \right] dx = \int_a^b \left[y_1 - y_2 \right] dx$$

Note: The area between two curves $x_1 = f(y)$ and $x_2 = g(y)$ from $y = c$ to $y = d$ is given by,

$$A = \int_c^d \left[f(y) - g(y) \right] dy = \int_c^d \left[x_1 - x_2 \right] dy$$

Examples

1) Find by integration the area between the curves $y = x$ and $y = x^2$.

2) Find the area enclosed between the two parabolas $y^2 = x$ and $x^2 = y$.

Examples for Tutorial

1) Find the area enclosed between the parabola $y = x^2$.and the line $y = 4$. $[A = 32/3]$

2) Find the area enclosed between two parabolas $y = 2x^2 - 5$ and $y = x^2 + 4$. $[A = 0.36]$

Volume of a solid of revolution

Consider an area bounded by the curve y = f(x), the ordinate at x = a and x =b and the x-axis. If we revolve the area about the x – axis, we get a solid of revolution.

1) If the region bounded by the curve y = f(x), the x – axis and the lines x = a and x = b is revolved about the x-axis, then the volume of solid of revolution is given by,

$$V = \pi \int_a^b y^2 dx = \pi \int_a^b [f(x)]^2 dx.$$

2) If the region bounded by the curve x = g(y), the y – axis and the lines y = c and y = d is revolved about the y-axis, then the volume of solid of revolution is given by,

$$V = \pi \int_a^b x^2 dx = \pi \int_a^b [g(y)]^2 dy.$$

Examples

1) Find the volume generated by revolving about the x-axis , the area bounded by 4y = 3x, x = 0 and x = 4. **[IoPE2012, B.T.E.'87]**
2) Find the volume generated by revolving semicircle about its bounded diameter. **[IoPE2008]**

Examples for Tutorial

1) The line x = a intersects the parabola $y^2 = 4ax$ in A. Find the volume of the solid generated by revolving the area bounded by the parabola, the line OX and the line x = a about the x-axis. **[B.T.E.'91]** $\left[V = 2\pi a^3 \text{ cubic units} \right]$

2) The loop of the curve $y^2 = x(x-1)^2$ is rotated about the x – axis. Find the volume of the solid generated. **[IoPE 2009]** $\left[V = \frac{5\pi}{12} \text{ cubic units} \right]$

3) Find the volume of the solid generated by revolving the area bounded by the curve y = sinx from x = 0 and $x = \pi/2$ about the X –axis. $\left[V = \frac{\pi^2}{4} \text{cubic units} \right]$