Complex Numbers

Definition of a complex number - - A number of the form z = x + iy, where x and y are real numbers and $i=\sqrt{-1}$ is called a complex number.

Note : 1) x is called the real part of z and it is denoted by R(z) or Re(z) and y is called the imaginary part of z and it is denoted by I(z) or Im(z).

2) $i=\sqrt{-1}$ $\therefore i^2=-1$

3) i ,j and k are imaginary numbers and $i=j=k=\sqrt{-1}$.

Equality of two complex numbers – Two complex numbers are said to be equal if and only if their real parts are equal and imaginary parts are equal .

Algebra of complex numbers – Let $z_1=a+ib$ and $z_2=c+id$ be two complex numbers where $a,b,c,d\in R$ and $i=\sqrt{-1}$. Then addition, subtraction, multiplication and division are performed as follows:

1) Addition:

$$z_1+z_2=(a+ib)+(c+id)$$
$$=(a+c)+i(b+d)$$

2) Subtraction:

$$z_1-z_2=(a+ib)-(c+id)$$
$$=(a-c)+i(b-d)$$

3) Multiplication:

$$z_1z_2=(a+ib)(c+id)$$
$$=a(c+id)+ib(c+id)$$
$$=ac+iad+ibc+i^2bd$$
$$=(ac-bd)+i(ad+bc) \quad [\because i^2=-1]$$

4) Division:

$$\frac{z_1}{z_2}=\frac{a+ib}{c+id}$$
$$=\frac{a+ib}{c+id}\times\frac{c-id}{c-id}=\frac{ac-iad+ibc-i^2bd}{c^2-i^2d^2}=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2} \quad [\because i^2=-1]$$

Conjugate of complex number- Two complex numbers which differ only in the sign of imaginary part are said to be conjugate of each other.

e.g. a + ib and a – ib are conjugates of each other.

Modulus (magnitude) and argument(amplitude) of a complex number:

1) If z = x + iy is a complex number, then the modulus of z is defined to be $\sqrt{x^2+y^2}$ and is denoted by r or $|z|$ or mod z.

$$\therefore r=|z|=\sqrt{x^2+y^2}$$

2) If z = x + iy is a complex number, then the argument of z is defined as $\tan^{-1}(y/x)$ and is denoted by θ or arg(z) or amp(z) .

$$\therefore \theta=\tan^{-1}(y/x)$$

Polar form of a complex number:

If z = x + iy is a complex number, then the polar form of complex number z is given by

$$z=r(\cos\theta+i\sin\theta)$$

Exponential form of a complex number:

If z = x + iy (Cartesian form) is a complex number, then $z = re^{i\theta}$ is called the exponential form of a complex number.
Note: 1) $e^{i\theta} = \cos\theta + i\sin\theta$, 2) $e^{-i\theta} = \cos\theta - i\sin\theta$

Examples: 1) If $z_1 = -1+2i$, $z_2 = -4+i$, find $z_1 z_2$. **[IoPE 2009]**

2) If $z_1 = -3+4i$, $z_2 = 5-3i$, find z_1z_2 and $\frac{1}{z_1}+\frac{1}{z_2}$. **[IoPE2012]**

3) If $z_1 = 6-2i$, $z_2 = 2-5i$, then find

a) $|2z_1-3z_2|$, b) $|3z_1+2z_2|$, c) $|z_1z_2|$ d) $|z_1/z_2|$, e) $|(z_1+z_2)(z_1-z_2)|$, f) $\left|\frac{z_1+z_2}{z_1-z_2}\right|$

Examples for Tutorial

1) If $z_1 = 5-2i$, $z_2 = 6+5i$, find $\left|\frac{z_1+z_2}{z_1-z_2}\right|$. **[IoPE2013,17]**

$$\left[\begin{array}{l} [\, z_1+z_2 = 11+3i,\quad z_1-z_2 = -1-7i,\ \frac{z_1+z_2}{z_1-z_2} = \frac{-31+74i}{50}, \\ \left|\frac{z_1+z_2}{z_1-z_2}\right| = \left|\frac{-31+74i}{50}\right| = \frac{1}{50}|-31+74i| = \sqrt{6437} = 80.2309 \end{array}\right]$$

2) If $z_1 = 4-5i$ and $z_2 = 3+7i$, find $|z_1 z_2|$ and $|z_1/z_2|$. **[IoPE 2015]**

$[\, z_1z_2 = 47+13i,\ |z_1z_2| = \sqrt{2378} = 48.7647$

$\frac{z_1}{z_2} = \frac{-23-43i}{58},\ \frac{|z_1|}{|z_2|} = \frac{48.7647}{58} = 0.8407]$

3) If $z = 1+3i$, evaluate z^2+2z+4. **[BTE2015]** $[-2+12i]$

Examples

Separate real and imaginary parts of the following complex numbers and also find its complex conjugate.
Or Express in the form of $x+iy$ and find the complex conjugate.

1) $\frac{15}{4+3j}$ **[IoPE 2012]** 2) $\left(\frac{1-i^3}{1+i^3}\right)^2$ $\left[\left(\frac{1-i^3}{1+i^3}\right)^2 = -1,\ C.C. = -1\right]$

Examples for Tutorial

Express in the form of x + iy and find the complex conjugate.

Examples	Answers
1) $\frac{3+\sqrt{-4}}{4-\sqrt{-9}}$	$\left[\frac{6}{25}+\frac{17}{25}i,\ C.C. = \frac{6}{25}-\frac{17}{25}i\right]$
2) $(2+3i)(1-4i)$ **[BTE2017]**	$[14-5i]$
3) $\frac{2+3i}{1-i}$ **[IoPE 2009]**	

4) $\dfrac{1-i^3}{2+i^3}$ **[IoPE 2016]** $\left[\dfrac{1-i^3}{2+i^3}=\dfrac{1+3i}{5}=\dfrac{1}{5}+\dfrac{3i}{5},\ \ C.C.=\dfrac{1}{5}-\dfrac{3i}{5}\right]$